

# A REAL K3 AUTOMORPHISM WITH MOST OF ITS ENTROPY IN THE REAL PART

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ABSTRACT. This article describes the first example of a real K3 surface  $X$  admitting a real automorphism  $f$  satisfying  $h_{top}(f, X(\mathbb{C})) < 2h_{top}(f, X(\mathbb{R}))$ . The example presented is a  $(2, 2, 2)$ -surface in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  defined by the vanishing set of  $(1 + x^2)(1 + y^2)(1 + z^2) + 10xyz - 2$ , first described by McMullen. Along the way, we develop an ad hoc shadowing lemma for  $C^2$  (real) surface diffeomorphisms, and apply it to estimate the location of a periodic point in  $X(\mathbb{R})$ . This result uses the GNU MPFR arbitrary precision arithmetic library in C and the Flipper computer program.

## CONTENTS

1.	Introduction	1
2.	Computing $h_{top}(f, X(\mathbb{C}))$	4
3.	Bounding $h_{top}(f, X_{10}(\mathbb{R}))$	7
	Appendices	17
A.	Proof of the shadowing lemma	17
B.	Miscellaneous	19
C.	Computer error estimates	22
D.	Tables	26
	References	27

## 1. INTRODUCTION

Let  $X$  be a K3 surface equipped with a real structure, and let  $f : X \rightarrow X$  be a *real automorphism* of  $X$ ; that is, a holomorphic diffeomorphism which commutes with the given antiholomorphic involution  $\sigma$  of  $X$ . The fixed point set of  $\sigma$ , denoted  $X(\mathbb{R})$ , is either empty or a (possibly disconnected) compact, real-analytic, totally real submanifold with real dimension two. Trivially,  $h_{top}(f, X(\mathbb{R})) \leq h_{top}(f, X(\mathbb{C}))$ . However, in his ICM survey ([4]), Cantat noted that there were no known examples for which  $h_{top}(f, X(\mathbb{C})) < 2h_{top}(f, X(\mathbb{R}))$ . The purpose of this paper is to provide such an example.

Before beginning, we recall a few related facts. Cantat proved in [2] that if  $Y$  is a connected compact complex surface equipped with an automorphism  $f$  of positive topological entropy, then  $Y$  is Kähler and one of the following holds:

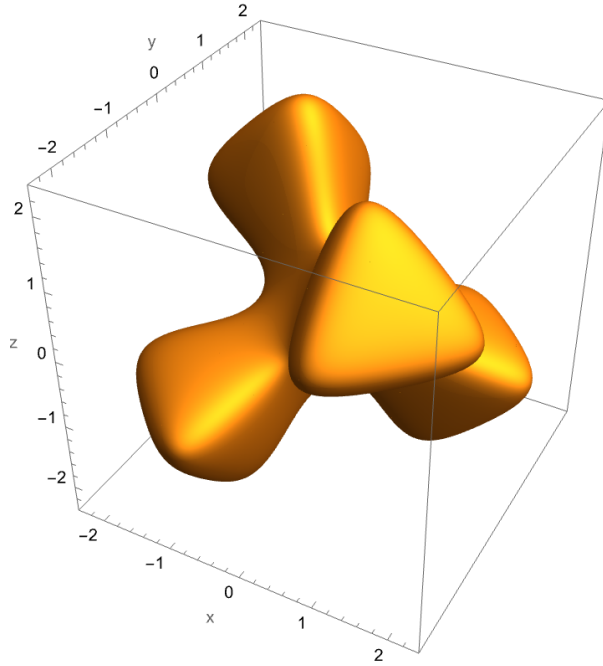
- (1) either  $Y$  is a non-minimal rational surface;
- (2) or  $Y$  is a K3 surface, a complex torus, or an Enriques surface;
- (3) or there exists a surface  $\tilde{Y}$  from case (2) and a holomorphic birational map  $\pi : Y \rightarrow \tilde{Y}$  such that  $\tilde{f} := \pi \circ f \circ \pi^{-1}$  is an automorphism of  $\tilde{Y}$ .

Given a surface  $Y$  with a real structure and a real automorphism  $f$  with positive entropy, we let  $\rho(f) := \frac{h_{top}(f, Y(\mathbb{R}))}{h_{top}(f, Y(\mathbb{C}))}$  denote the ratio of the real and complex entropies.

In case (3), the surface  $\tilde{Y}$  is obtained as a finite sequence of blow-ups along periodic orbits of  $f$ . Since the entropy of an automorphism of a Kähler manifold is determined by the spectral radius of its action on cohomology (see the Gromov-Yomdin theorem [9], [10], [23]),  $h_{top}(f, Y(\mathbb{C})) = h_{top}(\tilde{f}, \tilde{Y}(\mathbb{C}))$ . Moreover, even if some of the blow-ups are along an orbit in the real part, one can show  $h_{top}(f, Y(\mathbb{R})) = h_{top}(\tilde{f}, \tilde{Y}(\mathbb{R}))$ . Therefore, it's enough to focus on cases (1) and (2).

In the case that  $Y$  is a rational surface, Diller-Kim produced a family of examples for which  $\rho(f) = 1$  ([6], Theorem A). They also provided examples for which  $\frac{1}{2} < \rho(f) < 1$  ([6], Theorem B). To my knowledge, in the rational case, no examples are known for which  $\rho(f) = 0$ .

For K3 surfaces, it remains open whether  $\rho(f) = 1$  or  $\rho(f) = 0$  can occur. On one hand, Moncet proved that for any  $\varepsilon > 0$ , there exists a K3 surface admitting a real automorphism  $f$  with  $\rho(f) < \varepsilon$

FIGURE 1.  $X_{10}(\mathbb{R})$ , plotted in Mathematica

([18], Theorem 5.7). He also showed that if  $Y$  is a complex torus, then any real automorphism  $f$  of  $Y$  with positive entropy satisfies  $\rho(f) = \frac{1}{2}$  ([18], Proposition 4.4). The same holds when  $(Y, f)$  is a so-called Kummer example in the sense of [5]. For instance, a K3 Kummer example  $(Y, f)$  is obtained by taking  $Y$  to be the minimal regular model of the quotient of the standard complex torus  $\mathbb{C}^2/\Lambda$  by  $(x, y) \mapsto (-x, -y)$ , and taking  $f$  to be induced by a real hyperbolic automorphism of the torus.

Generally, a few strategies for bounding  $\rho(f)$  from below have emerged. While complex entropy can usually be computed explicitly using the Gromov-Yomdin theorem, the challenge becomes providing a lower bound on the real entropy. In [6], Diller-Kim bound the real entropy by understanding how  $f$  acts on the homology of the real part. In forthcoming work, Kim provides sharper lower bounds on  $\rho(f)$  for some automorphisms by analyzing homotopy growth rates. Alternatively, Moncet proved that  $\rho(f)$  is bounded below by an intrinsic property of the surface  $Y$  called its “concordance” ([18]). However, concordance is generally tricky to calculate. In the present work, we bound real entropy by computing the stretch factor of the mapping class of  $f$ , acting on the real surface with punctures along a real periodic orbit.

**1.1. Setup and Main Theorem.** Given nonzero  $A \in \mathbb{R}$ , consider

$$q_A(x, y, z) := (1 + x^2)(1 + y^2)(1 + z^2) + Axyz - 2.$$

Taking  $([x_0 : x], [y_0 : y], [z_0 : z])$  to be homogeneous coordinates for  $(\mathbb{P}^1)^3$ , one has  $q_A(x, y, z) = \tilde{q}_A([1 : x], [1 : y], [1 : z])$  for a tri-homogeneous polynomial  $\tilde{q}_A$ . Let  $X_A \subset (\mathbb{P}^1)^3$  denote the smooth real projective variety defined by the vanishing set of  $\tilde{q}_A$ .

Each  $X_A$  is an example of a so-called  $(2, 2, 2)$  surface in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , since  $q$  has degree two in each variable. Smooth surfaces in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  of tridegree  $(2, 2, 2)$  have been studied in several contexts including by Mazur ([15]) and Wang ([22]). One can check that every  $(2, 2, 2)$  surface  $S$  is K3. Indeed, a computation using the adjunction formula reveals the canonical bundle of  $S$  to be trivial. Moreover, the Lefschetz hyperplane theorem implies  $b_1(S) = 0$ . It follows that  $S$  is a K3 surface. The specific class of examples of the form  $X_A$  were considered by McMullen in [17].

There are three natural involutions of any  $(2, 2, 2)$ -surface which we recall here. Let  $\pi_i : (\mathbb{P}^1)^3 \rightarrow \mathbb{P}^1$  denote projection to the  $i$ th coordinate. For  $i \neq j$ , the map  $\pi_{ij} := (\pi_i, \pi_j) : (\mathbb{P}^1)^3 \rightarrow (\mathbb{P}^1)^2$  restricted to  $X_A$  has fibers generically containing exactly two points and otherwise containing one point or a rational curve. We define  $\sigma_k^A$  to be the involution of  $X_A$  which preserves each fiber of  $\pi_{ij}$  ( $i, j \neq k$ ) and swaps elements of a generic fiber. Since any birational self-map of a K3 surface extends to an automorphism,  $\sigma_k^A$  is an automorphism. The three involutions  $\sigma_i$  on any  $(2, 2, 2)$ -surface generate a free product  $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$  in  $\text{Aut}(X)$  (see [22]).

For each nonzero  $A \in \mathbb{R}$ , we consider the automorphism of  $X_A$  given by

$$f_A := \sigma_3^A \circ \sigma_2^A \circ \sigma_1^A.$$

In Section 2, we prove that  $h_{top}(f_A, X_A(\mathbb{C}))$  is positive and independent of  $A$ .

*Remark 1.1.* In [18], Moncet uses the continuity of topological entropy to prove that

$$\lim_{A \rightarrow 0} h_{top}(f_A, X_A(\mathbb{R})) = 0.$$

He applies a theorem of Cantat from [2] to obtain that  $h_{top}(f_A, X_A(\mathbb{C}))$  is positive and independent of  $A$ , implying  $\lim_{A \rightarrow 0} \rho(f_A) = 0$ . Our computation of  $h_{top}(f_A, X_A(\mathbb{C}))$  differs from Moncet's; he assumes that the fibers of  $\pi_{ij}$  do not contain any rational curves, which is not the case. Regardless,  $h_{top}(f_A, X_A(\mathbb{C}))$  remains positive and independent of  $A$ , so Moncet's conclusion holds.

*Remark 1.2.* It should be noted that the family of examples  $\{X_A\}$  is non-generic within all  $(2, 2, 2)$ -surfaces. Cantat proves in [2] that a generic  $(2, 2, 2)$ -surface has picard number 3, while we will see that each  $X_A$  has picard number at least 12. Moreover, Cantat proves that the automorphism  $\sigma_3 \circ \sigma_2 \circ \sigma_1$  on a generic  $(2, 2, 2)$ -surface has entropy  $\ln(9 + 4\sqrt{5}) \approx \ln 17.9443$ , while we will find  $h_{top}(f_A) \approx \ln 6.1393$ .

While the calculations in Section 2 apply for any nonzero  $A$ , in Section 3 we focus on the specific case  $A = 10$ . The real surface  $X_{10}(\mathbb{R})$  is homeomorphic to a sphere and completely contained in the affine part of  $X_{10}$  (see Figure 1).

**Theorem 1.3** (Main Theorem). *For  $X_A$  and  $f_A$  defined above:*

- (1)  $h_{top}(f, X_A(\mathbb{C})) \approx \ln 6.1393$  for every nonzero  $A \in \mathbb{R}$ ;
- (2) there exists a finite  $f_{10}$ -invariant set  $S$  such that the action of  $f_{10}^2$  on the complement  $X_{10}(\mathbb{R}) \setminus S$  has stretching factor  $\approx 8.1998$ .

As a consequence of (1) and (2),

$$\rho(f_{10}) \gtrsim \frac{\frac{1}{2} \ln 8.1998}{\ln 6.1393} > \frac{1}{2}.$$

**1.2. Outline of the paper.** Our computation of  $h_{top}(f, X_A(\mathbb{C}))$  in Section 2 relies on the Gromov-Yomdin theorem and working with explicit algebraic curves. In Section 3.2, we prove a checkable shadowing lemma for an arbitrary  $C^2$ -diffeomorphism of a (real) surface. In Sections 3.4 and 3.5, we apply the shadowing lemma to a pseudo-orbit of  $f_{10}$  on  $X_{10}(\mathbb{R})$ , with the help of a computer. The result is an  $f_{10}$ -periodic point  $x \in X_{10}(\mathbb{R})$  with period 10.

In Section 3.6, we compute the mapping class of  $f_{10}^2$  on  $X_{10}(\mathbb{R}) \setminus \{f^i(x) \mid i \in \mathbb{Z}\}$  in terms of Dehn half-twists. The mapping class representation of  $f_{10}^2$  is written explicitly in Section 3.6.2. From there, Flipper ([1]) is able to compute the stretch factor (dilatation) of  $f_{10}^2$ , yielding a lower bound on  $h_{top}(f_{10}, X_{10}(\mathbb{R}))$  (see [7]).

**1.3. Accompanying computer code.** The GitHub repository accompanying this article is K3entropy. See its README file for more details. All code is based on Curt McMullen's 'Orbits of Automorphisms of K3 Surfaces' (see [16]). Aside from Flipper, the three programs essential to the main theorem are:

- `periodicExact.c` – used in Section 3.5 – uses the GNU MPFR library (see [8]) to bound  $\max_i \|f_i^c(0)\|_2$  (see Lemma 3.2 for notation). Error bounds for this computation using interval arithmetic are provided in Appendix C.
- `derivativesExact.c` – used in Sections 3.5.1 and 3.5.2 – uses the GNU MPFR library to approximate each  $Df_i^c(0)$  (Table 3). It also estimates  $\mathcal{D}$  (see Section 3.4.1 for notation) and its first and second partials along a specific pseudo-orbit (Table 2).
- `arcs.c` – used in Section 3.6.2 and Figure 4 – plots the images of several arcs under  $f_{10}$ . The so-called *arc-data* of these image arcs (see Section 3.6.1) is readily obtained from these plots, and is also output automatically by `arcs.c`. The top image in Figure 4 is the output of `arcs.c`, where the input arcs connect successive elements of a certain 10-periodic orbit of  $f_{10}$  (see Section 3.6).

In Section 3.6.1, we provide an algorithm for realizing the mapping class of a homeomorphism of an  $n$ -punctured sphere as a product of Dehn half-twists. The algorithm is implemented in `mclass.c` and takes as input the arc-data obtained by `arcs.c`. In our case, the output of `mclass.c` can be verified by hand (see Figure 4).

Finally, we use Flipper to compute the stretch factor of the product of Dehn half-twists representing the mapping class of  $f_{10}^2$  on  $X_{10}(\mathbb{R}) \setminus \{f^i(x) \mid i \in \mathbb{Z}\}$ . The file `FlipperDilatation.ipynb` is a Jupyter Notebook which runs Flipper in SageMath ([21]) to compute the stretch factor.

1.4. **Notation.** Here we collect some notation used in the rest of the paper:

- $\|\cdot\|_2$  is the  $\ell^2$ -norm.
- $\|\cdot\|_\infty$  is the maximum norm.
- $\|\cdot\|_0$  is the norm on  $(\mathbb{R}^2)^n$  defined by  $\|(\vec{v}_1, \dots, \vec{v}_n)\|_0 := \max_{1 \leq i \leq n} \|\vec{v}_i\|_2$ .
- For  $M \in M_{m \times n}(\mathbb{R})$ ,  $\|M\|_{op}$  denotes the operator norm for  $M$  as a map  $(\mathbb{R}^n, \|\cdot\|_2) \rightarrow (\mathbb{R}^m, \|\cdot\|_2)$ .
- Given  $M \in M_{2n \times 2n}(\mathbb{R})$ ,  $\|M\|_{op,0}$  denotes the operator norm of  $M$  as a map  $((\mathbb{R}^2)^n, \|\cdot\|_0) \rightarrow ((\mathbb{R}^2)^n, \|\cdot\|_0)$ .
- For  $M \in M_{m \times n}(\mathbb{R})$ ,  $\|M\|_F$  is the Frobenius norm.
- For differentiable  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $D^2f$  is the Hessian matrix.
- For  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , let  $\|f\|_{C^0(U)} := \sup_{x \in U} \|f(x)\|_2$ .
- For  $f \in C^k(\mathbb{R}^n)$ , let  $\|f\|_{C^k(U), \infty} := \max_{|\beta| \leq k} \sup_{x \in U} |D^\beta f^i(x)|$ .
- For a  $C^k$ -function  $f = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , let  $\|f\|_{C^k(U), \infty} := \max_i \|f_i\|_{C^k(U), \infty}$ .
- For  $G : \mathbb{R}^n \rightarrow M_{\ell \times k}(\mathbb{R})$ , let  $\|g\|_{U, \star} := \sup_{x \in U} \|G(x)\|_\star$ , where  $\star \in \{op, F, \infty\}$ .

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## 2. COMPUTING $h_{top}(f, X(\mathbb{C}))$

**Notation 2.1.** For the rest of this section, we fix nonzero  $A \in \mathbb{R}$  and let  $X := X_A$  and  $f := f_A$ .

2.1. **Background.** The Gromov-Yomdin theorem implies that  $h_{top}(f, X(\mathbb{C})) = \ln R(f^*)$ , where  $R(f^*)$  denotes the spectral radius of  $f^* \in GL(H^*(X, \mathbb{C}))$ . In this section, we recall a few relevant facts about K3 surfaces (see, for example, [11]). By Hodge theory,  $X$  satisfies

$$\dim H^k(X, \mathbb{C}) = \begin{cases} 1 & k = 0, 4 \\ 22 & k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,  $f^*$  achieves its spectral radius on  $H^2(X, \mathbb{C})$ . Additionally, there exists an  $Aut(X)$ -invariant decomposition of  $H^2(X, \mathbb{C})$  into its Dolbeault cohomology classes, denoted

$$H^2(X, \mathbb{C}) = H^{2,0}(X, \mathbb{C}) \oplus H^{1,1}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C}).$$

Complex conjugation preserves  $H^{1,1}(X, \mathbb{C})$  and  $H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})$ , so both are spanned by elements of  $H^2(X, \mathbb{R})$ . By the Hodge Index Theorem, the intersection form  $\langle \alpha, \beta \rangle := \int_X \alpha \wedge \beta$  is positive-definite on  $(H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})) \cap H^2(X, \mathbb{R})$  and has signature  $(1, 19)$  on  $H^{1,1}(X, \mathbb{R}) := H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$ . Consequently,  $f^*$  achieves its spectral radius on  $H^{1,1}(X, \mathbb{R})$ . Define the Néron-Severi group by

$$NS(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})$$

in the sense that we consider the image of  $H^2(X, \mathbb{Z})$  in  $H^2(X, \mathbb{R})$  and intersect with  $H^{1,1}(X, \mathbb{R})$ . Then  $NS(X)$  is a discrete subgroup of  $H^2(X, \mathbb{R})$ . Since  $Aut(X)$  preserves  $H^2(X, \mathbb{Z})$  and the Hodge decomposition, it preserves  $NS(X)$ . Since  $X$  is a K3 surface, there exist  $Aut(X)$ -equivariant isomorphisms

$$D(X) \cong Pic(X) \cong NS(X)$$

where  $Pic(X)$  is the Picard group and  $D(X)$  is the free abelian group of algebraic curves considered up to linear equivalence. Moreover, under these isomorphisms,  $\langle \cdot, \cdot \rangle$  is carried to the intrinsic intersection forms on  $D(X)$  and  $Pic(X)$ . Since  $X$  is projective, the Picard group contains a line-bundle of positive self-intersection, so  $\langle \cdot, \cdot \rangle$  has signature  $(1, \rho(X) - 1)$  on each group, where  $\rho(X) := \dim Pic(X)$  is the picard number.

2.2. **Strategy.** We will find an explicit  $\sigma_i$ -invariant subspace  $W$  of  $D(X)$  on which  $\langle \cdot, \cdot \rangle$  is Minkowski. Then,  $R(f^*) = R(f^*|_W)$  will be directly computed.

**2.3. Computations on homology.** To that end, we consider several explicit algebraic curves in  $X$ . In homogeneous coordinates  $([x_0 : x], [y_0 : y], [z_0 : z])$  for  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , let

$$\begin{aligned} p_1 &:= \mathbb{P}^1 \times \{[0 : 1]\} \times \{[1 : i]\}, & p_2 &:= \mathbb{P}^1 \times \{[0 : 1]\} \times \{[1 : -i]\}, \\ p_3 &:= \mathbb{P}^1 \times \{[1 : i]\} \times \{[0 : 1]\}, & p_4 &:= \mathbb{P}^1 \times \{[1 : -i]\} \times \{[0 : 1]\}, \\ p_5 &:= \{[0, 1]\} \times \mathbb{P}^1 \times \{[1 : i]\}, & p_6 &:= \{[0, 1]\} \times \mathbb{P}^1 \times \{[1 : -i]\}, \\ p_7 &:= \{[1, i]\} \times \mathbb{P}^1 \times \{[0 : 1]\}, & p_8 &:= \{[1, -i]\} \times \mathbb{P}^1 \times \{[0 : 1]\}, \\ p_9 &:= \{[0 : 1]\} \times \{[1 : i]\} \times \mathbb{P}^1, & p_{10} &:= \{[0 : 1]\} \times \{[1 : -i]\} \times \mathbb{P}^1, \\ p_{11} &:= \{[1 : i]\} \times \{[0 : 1]\} \times \mathbb{P}^1, & p_{12} &:= \{[1 : -i]\} \times \{[0 : 1]\} \times \mathbb{P}^1. \end{aligned}$$

Each  $p_i$  is birationally equivalent to  $\mathbb{P}^1$ , and thus has self-intersection  $-2$  by the adjunction formula. Moreover, each  $p_i$  is the unique representative of  $[p_i] \in D(X)$ , so we use  $p_i$  and  $[p_i]$  interchangeably. A straightforward calculation, also performed in [20], yields

$$M := (\langle p_i | p_j \rangle)_i^j = \begin{pmatrix} -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \end{pmatrix}$$

and

$$S := (\langle p_i | \sigma_1 p_j \rangle)_i^j = \begin{pmatrix} -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}.$$

Diagonalizing  $M$  shows that  $\langle \cdot | \cdot \rangle$  has signature  $(1, 11)$  on  $W$ . Alternatively, we can realize an element of  $W$  with positive self-intersection in the following way. Each fiber of  $\pi_i : X_A \rightarrow \mathbb{P}^1$  is a curve given by the equation  $\{x_i = \alpha\}$ , all representing the same class in  $D(X)$ , which we denote by  $[c_i]$ . Notice  $\{x = [0 : 1]\} = p_5 \cup p_6 \cup p_9 \cup p_{10}$  so  $[c_1] = p_5 + p_6 + p_9 + p_{10}$ . Similarly,  $[c_2] = p_1 + p_2 + p_{11} + p_{12}$  and  $[c_3] = p_3 + p_4 + p_7 + p_8$ . So each  $[c_i]$  is contained in  $W$ . One can also check  $\langle [c_i], [c_j] \rangle = 2$  whenever  $i \neq j$  and vanishes otherwise. Thus  $[c_i] + [c_j]$ ,  $i \neq j$ , has positive self-intersection.

Define  $\tilde{\sigma}_1 : W \rightarrow W$  by  $\tilde{\sigma}_1 := M^{-1}S$  so that  $\tilde{\sigma}_1$  has the property

$$(2.2) \quad \langle \sigma_1^* w_1, w_2 \rangle = \langle \tilde{\sigma}_1 w_1, w_2 \rangle$$

for all  $w_1, w_2 \in W$ . One finds

$$\tilde{\sigma}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The fact that  $\tilde{\sigma}_1$  has integer entries and satisfied  $\tilde{\sigma}_1^2 = id$  was not a priori true, and suggests the following claim.

**Claim 2.3.**  $\tilde{\sigma}_1 = \sigma_1^*|_W$

*Proof.* Notice that  $\tilde{\sigma}_1^t M \tilde{\sigma}_1 = SM^{-1}S$ . Auspiciously, we saw that  $\sigma_1^2 = id$ , or equivalently,  $SM^{-1}S = M$ . Therefore  $\langle \cdot | \cdot \rangle$  is  $\tilde{\sigma}_1$ -invariant. Now  $\sigma_1^*$  is induced by an automorphism of  $X$  and thus also preserves  $\langle \cdot | \cdot \rangle$ . Therefore, for all  $w \in W$

$$\begin{aligned} \|\sigma_1^*(w) - \tilde{\sigma}_1(w)\|^2 &= \|\sigma_1^*(w)\|^2 + \|\tilde{\sigma}_1(w)\|^2 - 2\langle \sigma_1^*(w), \tilde{\sigma}_1(w) \rangle \\ &= \|\sigma_1^*(w)\|^2 + \|\tilde{\sigma}_1(w)\|^2 - 2\langle \tilde{\sigma}_1(w), \tilde{\sigma}_1(w) \rangle \\ &= \|w\|^2 + \|w\|^2 - 2\|\tilde{\sigma}_1(w)\|^2 \\ &= \|w\|^2 + \|w\|^2 - 2\|w\|^2 \\ &= 0. \end{aligned}$$

where in the second equality we use 2.2 and in subsequent equalities we use that  $\tilde{\sigma}_1, \sigma_1^*$  preserve  $\langle \cdot | \cdot \rangle$ . But for all  $1 \leq i, j \leq 12$ ,

$$\begin{aligned} \langle p_i, \tilde{\sigma}_1 p_j \rangle &= e_i^t M \tilde{\sigma}_1 e_j \\ &= e_i^t M M^{-1} S e_j \\ &= e_i^t S e_j = \langle p_i, \sigma_1 p_j \rangle \end{aligned}$$

and therefore  $\sigma_1^*(w) - \tilde{\sigma}_1(w) \in W^\perp$  for all  $w \in W$ . We conclude  $\sigma_1^*(w) = \tilde{\sigma}_1(w)$  since  $\langle \cdot | \cdot \rangle$  is negative definite on  $W^\perp$ .  $\square$

Let's check that  $\sigma_1^*$  does what we expect to  $[c_1], [c_2], [c_3]$ . Well,  $\sigma_1^*$  should fix  $[c_2] = p_1 + p_2 + p_{11} + p_{12}$  and  $[c_3] = p_3 + p_4 + p_7 + p_8$ ; our matrix does this. Moreover,

$$(2.4) \quad \pi_{23}^{-1}(\pi_{23}(c_1)) = \sigma_1(c_1) \cup c_1 \cup p_1 \cup p_2 \cup p_3 \cup p_4.$$

Now  $\pi_{23}(c_1) \subset \mathbb{P}^1 \times \mathbb{P}^1$  has bidegree  $(2, 2)$  and is therefore contained in the class  $2[\{\alpha\} \times \mathbb{P}^1] + 2[\mathbb{P}^1 \times \{\alpha\}]$ . Thus 2.4 becomes

$$2[c_2] + 2[c_3] = \sigma_1^*[c_1] + [c_1] + p_1 + p_2 + p_3 + p_4,$$

i.e.,

$$\sigma_1^*(p_5 + p_6 + p_9 + p_{10}) = 2(p_{11} + p_{12} + p_7 + p_8) + p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_9 - p_{10}$$

which our matrix also satisfies. A similar computation is performed in [19]. Using symmetry, one computes

$$\sigma_2^*|_W = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\sigma_3^*|_W = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$f^*|_W = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 3 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now,  $R(f^*) = R(f^*|_W)$  since  $\langle \cdot | \cdot \rangle$  is Minkowski on  $W$ . Moreover, since  $f^*$  preserves  $\langle \cdot | \cdot \rangle$ , the characteristic polynomial  $p(x)$  for  $f^*|_W$  is a *Salem polynomial*, meaning it has even degree, real coefficients, at most two roots without unit size, and satisfies  $p(0) = 1$ . We find

$$p(x) = 1 - 8x + 15x^2 - 24x^3 + 14x^4 - 8x^5 - 5x^6 - 8x^7 + 14x^8 - 24x^9 + 15x^{10} - 8x^{11} + x^{12},$$

which factors as a product of a polynomial with all unit roots and  $1 - 5x - 6x^2 - 5x^3 - 6x^4 - 5x^5 + x^6$ . Then  $R(f^*|_W)$  is the unique real root of

$$1 - 5x - 6x^2 - 5x^3 - 6x^4 - 5x^5 + x^6$$

with modulus  $> 1$ , which is  $\approx 6.1393$ . By the Gromov-Yomdin theorem,

$$(2.5) \quad h_{top}(f, X(\mathbb{C})) \approx \ln 6.1393.$$

*Remark 2.6.* The above calculation shows that  $h_{top}(f^A)$  is independent of  $A \in \mathbb{R}^*$ .

### 3. BOUNDING $h_{top}(f, X_{10}(\mathbb{R}))$

**Notation 3.1.** For the rest of the article, we let  $X := X_{10}$ ,  $f := f_{10}$  and  $q := q_{10}$ .

**3.1. Strategy.** Given a surface  $S = S_{g,n}$ , recall that the mapping class group of  $S$  is

$$\text{MOD}^\pm(S) := \text{Homeo}(S)/\text{Homeo}_0(S),$$

noting that we allow for orientation-reversing homeomorphisms. Let  $\text{MOD}^+(S)$  denote the index-2 subgroup of  $\text{MOD}^\pm(S)$  consisting of classes coming from orientation-preserving homeomorphisms. Given a homeomorphism  $h$  of  $S$ , we let  $[h] \in \text{MOD}^\pm(S)$  denote its mapping class.

In the following sections, we will approximate an  $f$ -periodic point  $x \in X(\mathbb{R})$  of order 10. Then we examine the action of  $f$  on  $S_{0,10} \cong X(\mathbb{R}) \setminus \{f^i(x)\}_{i=0}^9$ . Our goal is to represent  $[f^2]$  as a product of Dehn half-twists in  $\text{MOD}^+(S_{0,10})$ . Such a representation allows Flipper to show that  $[f^2]$  is a pseudo-Anosov mapping class and compute its stretch factor, denoted  $\lambda([f^2])$ . Since each  $\sigma_i$  is orientation-reversing on  $X(\mathbb{R})$ ,  $f$  is orientation-reversing, which is why we consider  $f^2$ . Recall that a pseudo-Anosov homeomorphism  $\varphi$  has minimal entropy in its mapping class, and moreover  $h_{\text{top}}(\varphi) = \ln \lambda([\varphi])$  (see [7]). Therefore,

$$h_{\text{top}}(f) = \frac{1}{2} h_{\text{top}}(f^2) \geq \frac{1}{2} \ln \lambda([f^2]).$$

Now, we turn to approximating such a periodic point  $x$ . Since  $f$  has positive entropy on  $X(\mathbb{C})$ , a classical theorem of Katok implies that the number of  $n$ -periodic points in  $X(\mathbb{C})$  grows at least exponentially in  $n$  (see [13]). However, finding explicit periodic points remains a challenge, especially those in  $X(\mathbb{R})$ . To that end, we prove an ad hoc shadowing lemma for  $C^2$  surface diffeomorphisms, and then use a computer to exhibit a pseudo-orbit  $(x_i)$  with sufficient recurrence and hyperbolicity to admit a nearby periodic point.

**3.2. A shadowing lemma.** The following shadowing lemma can be thought of as an ad hoc version of the Anosov shadowing lemma, where we only care about data along a given pseudo-orbit. The lemma requires some type of hyperbolicity in coordinates along the pseudo-orbit, control of the second derivative in coordinates along the pseudo-orbit, and high recurrence. The benefit is that all constants can be computed explicitly.

**Lemma 3.2.** *Let  $M$  be a smooth surface. Let*

- $h : M \rightarrow M$  a  $C^2$  diffeomorphism
- $x_0, \dots, x_{n-1} \in M$
- $\{\varphi_i : V_i \rightarrow M\}$  a set of  $C^2$  charts such that  $x_i = \varphi_i(0)$ .

Assume  $h(x_i) \in \varphi_{i+1 \pmod n}(V_{i+1 \pmod n})$  for all  $0 \leq i < n$ . Define

- $h_i^c := \varphi_{i+1 \pmod n}^{-1} \circ h \circ \varphi_i : U_i \rightarrow \mathbb{R}^2$  for  $0 \leq i < n$ , where

$$U_i := V_i \cap ((\varphi_{i+1 \pmod n}^{-1} \circ h \circ \varphi_i)^{-1}(V_{i+1 \pmod n}))$$

- $L_i := (Dh_i^c)_0$  and

$$L := \begin{pmatrix} 0 & L_0 & 0 & \dots & 0 \\ \vdots & 0 & L_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & L_{n-2} \\ L_{n-1} & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Assuming  $L - I_{20}$  is invertible (or equivalently  $L_{n-1}L_{n-2} \cdots L_0 - I_2$  is invertible), let  $C := \|(L - I_{20})^{-1}\|_{\text{op},0}$  (recalling Notation 1.4). If there exists  $\delta > 0$  such that

- (1) for all  $i$ ,  $B_{12C\delta}(0) \subset U_i$
- (2) for all  $i$ ,

$$12C\delta < \frac{1}{C} \frac{1}{16 \|D^2 h_i^c\|_{C^0(B_{12C\delta}(0)), \infty}}$$

- (3) for all  $i$ ,  $\|h_i^c(0)\|_2 < \delta$

then there exists  $h$ -periodic  $p \in M$  with period  $n$  and  $h^i(p) \in \varphi_i(B_{6C\delta}(0))$ .

Our proof mirrors Katok-Hasselblatt's proof of the Anosov shadowing lemma in [14], Theorem 6.4.15. We delay it until Appendix A.

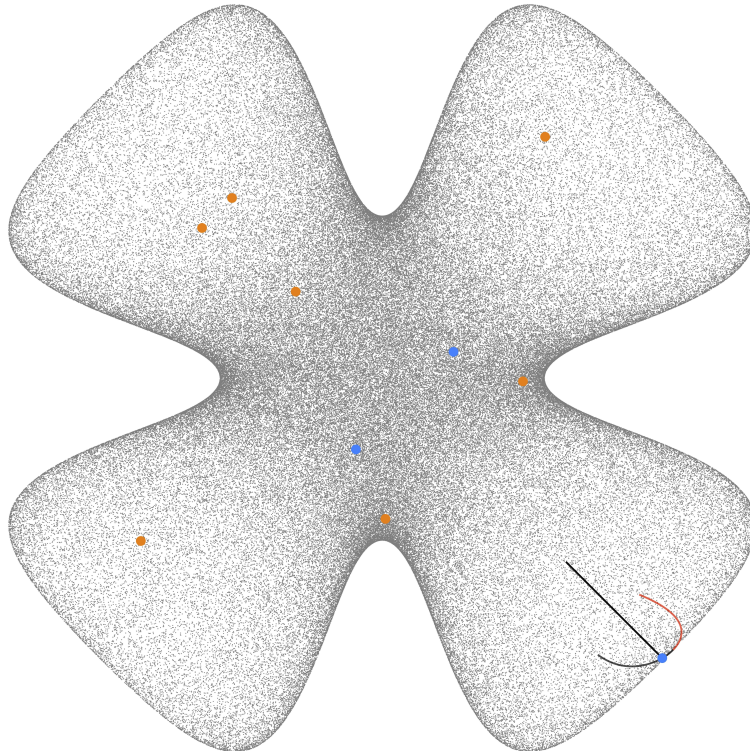


FIGURE 2. The pseudo-orbit  $x_i = \tilde{\varphi}_i(a_i, b_i)$  from Table 1 plotted on  $X(\mathbb{R})$ , and projected into the  $z$ - $x$  plane. Blue dots denote points on the top sheet and orange the bottom sheet. The curve  $C := \{x = -z\} \cap X$  can be seen intersecting  $f^5(C)$  near  $x_0$ . This figure is obtained using a program developed by C. McMullen.

**3.3. Where to look for pseudo-orbits.** Let  $\rho : (\mathbb{P}^1)^3 \rightarrow (\mathbb{P}^1)^3$  denote reflection through  $\{x = -z\}$ . Notice that  $\tilde{q}$  is  $\rho$ -invariant, so  $\rho(X) = X$ . Furthermore,  $C := \{x = -z\} \cap X$  is cut out by an equation of bidegree  $(4, 2)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Consequently,  $C$  has arithmetic genus  $3 = (4 - 1)(2 - 1)$  and thus self-intersection  $4 = 3 \cdot 2 - 2$  by the adjunction formula for a K3 surface. A simple computation reveals that  $\rho \circ f = f^{-1} \circ \rho$ . Therefore, any  $p \in C \cap f^n(C)$  is  $f$ -periodic of order dividing  $2n$ .

Recall that there is a unique projective class in the isotropic cone of  $\langle \cdot | \cdot \rangle$  in  $H^{1,1}(X, \mathbb{R})$  that is the attracting fixed point for the action of  $f$ . Let  $v_f^+$  denote a representative of this class. Similarly,  $f$  has a repelling projective class from which we pick a representative  $v_f^-$  satisfying  $\langle v_f^+, v_f^- \rangle = 1$ . Then

$$\lim_{n \rightarrow \infty} e^{-nh_{top}(f)} \langle C, f^n(C) \rangle = \langle C, v_f^- \rangle \cdot \langle C, v_f^+ \rangle$$

(see [3], section 2.3). Since  $[C]$  has positive self-intersection,  $\langle C, v_f^- \rangle$  and  $\langle C, v_f^+ \rangle$  are positive. Thus, the number of  $f$ -periodic points of order  $2n$  contained in  $C$  grows like  $e^{nh_{top}(f)}$ . A priori, it might be the case that none of these periodic points are contained in  $X(\mathbb{R})$ . However, one can plot  $C$  and  $f^n(C)$  and see many intersections in  $X(\mathbb{R})$ ; in the following section, we will input into Lemma 3.2 a pseudo-orbit coming from an approximate intersection point of  $C$  and  $f^5(C)$  in  $X(\mathbb{R})$ .

It's worth noting that one can calculate the intersection of  $C$  with each  $p_i$  and use the intersection matrix from Section 2.3 to arrive at  $\langle proj_W[C], proj_W[C] \rangle = 4$ . But  $\langle [C], [C] \rangle = 4$  and  $W^\perp$  is negative-definite, so in fact  $[C] \in W$ . Therefore, our formula for the action of  $f^*$  on  $W$  allows us to explicitly compute  $\langle C, f^n(C) \rangle$  for any given  $n$ .

**3.4. A specific pseudo-orbit.** By the previous section, any intersection between  $C$  and  $f^n(C)$  in  $X(\mathbb{R})$  will be a  $2n$ -periodic point. Figure 2 shows such an intersection when  $n = 5$ . In the figure, we selected a point near the intersection and plotted the first ten points in its orbit. This will serve as our pseudo-orbit. In the following section, we define the  $\varphi_i$  used in Lemma 3.2.

**3.4.1. Charts.** Let  $\mathcal{D}(x, y)$  denote the discriminant of  $q$  with respect to  $z$ , i.e.

$$\mathcal{D}(x, y) := 10^2 x^2 y^2 + 8(1 + x^2)(1 + y^2) - 4(1 + x^2)^2(1 + y^2)^2.$$

$i$	$\tilde{\varphi}_i$	$a_i$	$b_i$
0	$\Psi_1^-$	1.041643093944314148360673792017	1.726895448754858426328854724474
1	$\Psi_3^-$	-0.439586738044637984442175311821	0.555943953085459715621476373770
2	$\Psi_2^-$	1.111402054756051352317454938205	-0.926435350008842162121508383319
3	$\Psi_1^-$	-0.328869789067645570794396144391	0.867950394543647373540816310310
4	$\Psi_2^-$	1.488818954806569814700993326668	1.004464450964796276608907444033
5	$\Psi_2^-$	0.533829900932504729554816817729	-0.533829900932504729554816817729
6	$\Psi_2^-$	-1.004464450964796276608907444033	-1.488818954806569814700993326668
7	$\Psi_3^+$	-0.867950394543647373540816310310	-0.328869789067645570794396144391
8	$\Psi_2^-$	0.926435350008842162121508383319	-1.111402054756051352317454938205
9	$\Psi_1^+$	0.555943953085459715621476373770	0.439586738044637984442175311821

TABLE 1. A pseudo-orbit

A straightforward computation reveals  $\mathcal{D}(x, y) = (\partial_z q(x, y, z))^2$ . Therefore, since  $X$  is a smooth variety, we cannot have simultaneous vanishing of  $\mathcal{D}(x, y), \mathcal{D}(x, z), \mathcal{D}(y, z)$  on  $X$ . Let

$$p_{\pm}(x, y) := \frac{-10xy \pm \mathcal{D}(x, y)^{\frac{1}{2}}}{2(1+x^2)(1+y^2)}$$

where  $z \mapsto z^{\frac{1}{2}}$  is taken to be the principal branch of the square root. Then the maps

$$\begin{aligned} \Psi_1^{\pm}(x, y) &:= ([1 : p_{\pm}(y, z)], [1 : y], [1 : z]) \\ \Psi_2^{\pm}(x, y) &:= ([1 : x], [1 : p_{\pm}(x, z)], [1 : z]) \\ \Psi_3^{\pm}(x, y) &:= ([1 : x], [1 : y], [1 : p_{\pm}(a_i, b_i)]) \end{aligned}$$

are holomorphic coordinates defined on

$$\{(x, y) \in \mathbb{C}^2 \mid x, y \notin \{\pm i\} \text{ and } \mathcal{D}(x, y) \neq 0\}.$$

We consider the collection of points and charts enumerated in Table 1. For each  $i$ , define

$$x_i := \tilde{\varphi}_i(a_i, b_i).$$

Moreover, let

$$\varphi_i(\zeta, \gamma) := \tilde{\varphi}_i(\zeta + a_i, \gamma + b_i)$$

so that the  $\varphi_i$ 's and  $x_i$ 's are as in Lemma 3.2. Here

$$V_i = \{(x, y) \in \mathbb{R}^2 \mid \mathcal{D}(x + a_i, y + b_i) > 0\}$$

and

$$U_i = V_i \cap (f_i^c)^{-1}(V_{i+1 \pmod{10}}).$$

We emphasize that the *exact* values of the  $a_i, b_i$  are displayed in Table 1. We will see that this pseudo-orbit has enough recurrence and hyperbolicity to apply Lemma 3.2. One could choose  $a_i, b_i$  more carefully to get even more recurrence, with a negligible change in the first and second derivatives of  $f_i^c$ .

**3.5. Applying the shadowing lemma.** Recall the notation used in Lemma 3.2. Fix  $\varepsilon' := 10^{-18}$ . In Section 3.5.1, we prove that  $B_{\varepsilon'}(0) \subset U_i$  and  $\max_i \|D^2 f_i^c\|_{C^0(B_{\varepsilon'}(0)), \infty} < 1.4 \cdot 10^{14}$ . Moreover, in Section 3.5.2 we prove that  $C < 21$  for our choices of  $\varphi_i$ . Therefore, by Lemma 3.2, if  $\|f_i^c(0)\|_2 \leq \delta$  for all  $i$  where

$$\delta < \min \left( \frac{\varepsilon'}{12C}, \frac{1}{12C} \frac{1}{16C} \cdot \frac{1}{1.4 \cdot 10^{14}} \right) \approx 3.97 \cdot 10^{-21},$$

then there exists an  $f$ -periodic point  $x \in X(\mathbb{R})$  of period 10 and  $\|\varphi_i^{-1}(f^i(x))\|_2 < 6C\delta$ .

In `K3entropy`, `periodicExact.c`, we find that  $\|f_i^c(0)\|_2 < 10^{-29}$  for all  $i$ . Sufficient accuracy of the computer computation is guaranteed using the GNU MPFR library in C; see Appendix C for a discussion of computational errors. Therefore, there exists  $x \in X(\mathbb{R})$  such that  $f^{10}(x) = x$  and  $\|\varphi_i^{-1}(f^i(x))\|_2 < 6 \cdot 21 \cdot 10^{-29} < 10^{-26}$ .

3.5.1. *Bounding*  $\|D^2 f_i^c\|_{C^0(B_\varepsilon(0)),\infty}$  for small  $\varepsilon$ . Define

$$\begin{aligned} K &:= \max_i |\mathcal{D}(a_i, b_i)| \\ R &:= \max_i \|D\mathcal{D}(a_i, b_i)\|_\infty \\ M &:= \max_i \|D^2\mathcal{D}(a_i, b_i)\|_\infty. \end{aligned}$$

In `K3entropy`, `derivativesExact.c`, we use the GNU MPFR library to estimate  $\mathcal{D}(a_i, b_i)$  and its first and second partials at  $(a_i, b_i)$  for each  $i$ . One could produce precise computational error estimates analogous to those in Appendix C to show that the errors are small, but we will see that we don't need much accuracy. Indeed, from Table 2, we assume only that  $K < 114$ ,  $R < 163$ ,  $M < 441$ . Using the approximate values for  $\min_i \mathcal{D}(a_i, b_i)$  and  $R$ , along with the mean value theorem, we find that  $\mathcal{D} > 0$  on  $B_{10^{-3}}(a_i, b_i)$  for all  $i$ . In other words,  $B_{10^{-3}}(0) \subset V_i$  for all  $i$ .

Since each  $\varphi_i^{-1}$  is an everywhere-defined projection map, in what follows we consider  $f_i^c$  to have domain  $V_i$  instead of  $U_i$ .

**Lemma 3.3.** *Fix  $\varepsilon := 10^{-5}$ . Then  $\max_i \|Df_i^c\|_{C^1(B_\varepsilon(0)),\infty} < 1.4 \cdot 10^{14}$ .*

*Proof of Lemma 3.3.* In Lemma B.4 of the Appendix, we bound  $\|Df\|_{C^0(X(\mathbb{R})),\infty}$  and  $\|D^2 f\|_{C^0(X(\mathbb{R})),\infty}$ , where  $f$  is considered to be a function from a neighborhood of  $X(\mathbb{R}) \subset \mathbb{R}^3$  into  $\mathbb{R}^3$ . The remaining piece is to bound  $\|Dp\|_{C^1(B_\varepsilon(a_i, b_i)),\infty}$  and  $\|D^2 p\|_{C^1(B_\varepsilon(a_i, b_i)),\infty}$ , and apply the chain rule. We proceed via a series of short claims.

**Claim 3.4.**  $\max_i \max_{|\beta|=2} \sup_{B_\varepsilon(a_i, b_i)} |D^\beta D| \leq \sqrt{2} \cdot 20000 \cdot \varepsilon + R$ .

**Claim 3.5.**  $\max_i \max_{|\beta|=1} \sup_{B_\varepsilon(a_i, b_i)} |D^\beta \mathcal{D}| \leq \sqrt{2}(\sqrt{2} \cdot 20000 \cdot \varepsilon + R)\varepsilon + M$ .

**Claim 3.6.**  $\max_i \sup_{B_\varepsilon(a_i, b_i)} |\mathcal{D}| \leq \sqrt{2}(\sqrt{2}(\sqrt{2} \cdot 20000 \cdot \varepsilon + R)\varepsilon + M)\varepsilon + K$ .

Claim 3.4 follows from Lemma B.3 of the Appendix and the mean value theorem, with the  $\sqrt{2}$  picked up as a dimensionality constant. Claim 3.5 similarly follows from Claim 3.4, and Claim 3.6 from Claim 3.5.

**Claim 3.7.** *For all  $i$ ,*

$$\max_i \|D^2 p\|_{C^0(B_\varepsilon(a_i, b_i)),\infty} \leq 2.5 \cdot 10^4$$

and

$$\max_i \|Dp\|_{C^0(B_\varepsilon(a_i, b_i)),\infty} \leq 120.$$

From Claims 3.4, 3.5, 3.6 and using  $\varepsilon$  small, we obtain  $\max_i \sup_{B_\varepsilon(a_i, b_i)} |\mathcal{D}| < K + 1 = 115$  and  $\|D\mathcal{D}\|_{C^1(B_\varepsilon(a_i, b_i)),\infty} < \max(R, M) + 1 = 442$ . Then Claim 3.7 follows by B.5 of the Appendix.

To conclude the proof of Lemma 3.3, we examine the action of  $f_i^c$  on the space of 2-jets from  $\mathbb{R}^2$  to  $\mathbb{R}$  (see Appendix B.1). Indeed, this action is given by a matrix-valued map  $J^2(f_i^c) : B_\varepsilon(0) \rightarrow M_{5 \times 5}(\mathbb{R})$  whose entries contain the first and second partials of  $f_i^c$ . Therefore we have the bound

$$\|Df_i^c\|_{C^1(B_\varepsilon(0)),\infty} \leq \|J^2(f_i^c)\|_{C^0(B_\varepsilon(0)),\infty}.$$

Jets behave nicely under composition, meaning  $J_x^2(f_i^c) = J_x^2(\varphi_i) \cdot J_{\varphi_i(x)}^2(f) \cdot J_{f \circ \varphi_i(x)}^2(\varphi_{i+1}^{-1})$ , so we have

$$(3.8) \quad \|J^2(f_i^c)\|_{B_\varepsilon(0),op} \leq \|J^2(\varphi_i)\|_{B_\varepsilon(0),op} \cdot \|J^2(f)\|_{X(\mathbb{R}),op} \cdot \|J^2(\varphi_{i+1}^{-1})\|_{\varphi_{i+1}(B_\varepsilon(0)),op}.$$

But  $\varphi_{i+1}^{-1}$  is a projection, so  $J_y^2(\varphi_{i+1}^{-1})$  is an inclusion for all  $y$ , and thus  $\|J_y^2(\varphi_{i+1}^{-1})\|_{op} = 1$ . Moreover, by B.6,  $J^2(f)$  satisfies

$$\|J^2(f)\|_{X(\mathbb{R}),\infty} \leq \max(2\|Df\|_{C^0(X(\mathbb{R})),\infty}^2, \|D^2 f\|_{C^0(X(\mathbb{R})),\infty}).$$

By Lemma B.4,  $\|Df\|_{C^0(X(\mathbb{R})),\infty} \leq 9 \cdot 10^3$  and  $\|D^2 f\|_{C^0(X(\mathbb{R})),\infty} \leq 9^2(2 \cdot 10^2)^3$ . From this, we calculate  $\|J^2(f)\|_{X(\mathbb{R}),\infty} \leq 9^2(2 \cdot 10^2)^3$ . Note that  $J_{x_0}^2(f) \in M_{9 \times 9}(\mathbb{R})$  for all  $x_0$ , so picking up a dimensionality constant yields

$$(3.9) \quad \|J^2(f)\|_{X(\mathbb{R}),op} \leq 3 \cdot 9^2(2 \cdot 10^2)^3.$$

Finally,  $J^2(\varphi_i)$  satisfies

$$\|J^2(\varphi_i)\|_{C^0(B_\varepsilon(0)),\infty} \leq \max(2\|Dp\|_{C^0(B_\varepsilon(a_i, b_i)),\infty}^2, \|D^2 p\|_{C^0(B_\varepsilon(a_i, b_i)),\infty}),$$

and by Claim 3.7, the RHS is bounded above by  $\max(2 \cdot 120^2, 2.5 \cdot 10^4) < 3 \cdot 10^4$ . Since  $J_{x_0}^2(\varphi_i) \in M_{5 \times 9}(\mathbb{R})$  for all  $x_0$ , we obtain

$$(3.10) \quad \|J^2(\varphi_i)\|_{B_{\varepsilon}(0), op} < \sqrt{5} \cdot 3 \cdot 10^4 < 7 \cdot 10^4.$$

Inserting 3.9 and 3.10 into 3.8 yields

$$\begin{aligned} \|Df_i^c\|_{C^1(B_{\varepsilon}(0))} &\leq \|J^2(f_i^c)\|_{\infty} \\ &\leq \|J^2(f_i^c)\|_{op} \\ &< 3 \cdot 9^2(2 \cdot 10^2)^3 \cdot 7 \cdot 10^4 \\ &< 1.4 \cdot 10^{14}, \end{aligned}$$

proving Lemma 3.3. □

**Lemma 3.11.** *Fix  $\varepsilon' := 10^{-18}$ . Assume  $\max_i \|f_i^c(0)\|_2 < \frac{1}{2}10^{-3}$ . Then  $B_{\varepsilon'}(0) \subset U_i$  for all  $i$ .*

*Proof of Lemma 3.11.* By Lemma 3.3, we have

$$(3.12) \quad \max_i \|Df_i^c\|_{C^0(B_{\varepsilon'}(0)), \infty} < 1.4 \cdot 10^{14},$$

which implies  $\max_i \|Df_i^c\|_{C^0(B_{\varepsilon'}(0)), op} < \sqrt{2} \cdot 1.4 \cdot 10^{14}$ . We chose  $\varepsilon'$  such that  $\varepsilon' \cdot \sqrt{2} \cdot 1.4 \cdot 10^{14} < \frac{1}{2}10^{-3}$ . Then for each  $i$ , the image of the  $\varepsilon'$ -ball under  $f_i^c$  satisfies  $f_i^c(B_{\varepsilon'}(0)) \subset B_{\frac{1}{2}10^{-3}}(f_i^c(0))$ . By assumption,  $\|f_i^c(0)\|_2 < \frac{1}{2}10^{-3}$ , so it follows that  $f_i^c(B_{\varepsilon'}(0)) \subset B_{10^{-3}}(0) \subset V_{i+1 \pmod{10}}$ , i.e.  $B_{\varepsilon'}(0) \subset U_i$ . □

3.5.2. *Bounding  $C$ .* Recall the definitions of  $L_i, L$  from Lemma 3.2. We hope to bound

$$C := \|(L - I)^{-1}\|_{op, 0} = \frac{1}{\sigma_n(L - I)}.$$

Using the GNU MPFR library in C, implemented in K3entropy, `derivativesExact.c`, we estimate each  $L_i$  with high accuracy. That is, the program produces  $\tilde{L}_i$  such that  $\|L_i - \tilde{L}_i\|_{\infty} < r$  where  $r$  is small. Again, one could produce very precise error estimates analogous to those in Appendix C, but we will find in this section that we can tolerate some inaccuracy. The  $\tilde{L}_i$ 's, rounded to four decimals, are displayed in Table 3. Since  $f|_{X(\mathbb{R})}$  is volume-preserving (which is true for any real automorphism of a K3 surface), the product  $L_9 L_8 \dots L_0$  should have determinant 1. One can verify that  $\tilde{L}_9 \tilde{L}_8 \dots \tilde{L}_0$  has determinant  $\approx 1$ .

Letting  $\tilde{L}$  be the analog of  $L$ , we have  $\|L - \tilde{L}\|_{\infty} < r$ . Therefore  $\sup_{\|v\|_0=1} \|L(v) - \tilde{L}(v)\|_0 \leq r\sqrt{2}$ , and, in particular,  $|\sigma_n(L - I) - \sigma_n(\tilde{L} - I)| \leq r\sqrt{2}$ . So

$$\begin{aligned} \left| \|(L - I)^{-1}\|_{op, 0} - \|(\tilde{L} - I)^{-1}\|_{op, 0} \right| &= \left| \frac{1}{\sigma_n(L - I)} - \frac{1}{\sigma_n(\tilde{L} - I)} \right| \\ &\leq r\sqrt{2} \cdot \left( \frac{1}{\sigma_n(\tilde{L} - I) - r\sqrt{2}} \right)^2. \end{aligned}$$

Mathematica computes  $\|(\tilde{L} - I)^{-1}\|_F \approx 19.8966$  (see K3entropy, `MatrixEstimates.nb`), so  $\sigma_n(\tilde{L} - I) \geq \frac{1}{20}$ . Thus

$$\|(L - I)^{-1}\|_{op, 0} \leq 20 + r\sqrt{2} \frac{1}{\frac{1}{20} - r\sqrt{2}}.$$

Using even a conservative estimate for  $r$ , say  $r = 10^{-2}$ , we find

$$\|(L - I)^{-1}\|_{op, 0} < 21.$$

Therefore  $C < 21$ .

**3.6. Computing the mapping class of  $f^2$  on a sphere with punctures.** In Lemma B.1, we demonstrate that the component of  $\mathbb{R}^3 \setminus X(\mathbb{R})$  containing the origin is star-shaped. This allows us to easily isotope  $X(\mathbb{R})$  to the unit sphere  $S^2 \subset \mathbb{R}^3$ . In an abuse of notation, we still use  $f$  to denote the associated homeomorphism of  $S^2$ . Next, we stereographically project through  $\gamma := (0, 0, 1)$  to the  $x$ - $y$  plane, and rescale to the open unit disk  $\mathring{D} \subset \mathbb{R}^2$  for convenience; denote by  $\psi_\gamma : S^2 \setminus \{\gamma\} \rightarrow \mathring{D}$  the composition of projection and rescaling. Since  $\gamma \notin \{f^k(x)\}_{k \in \mathbb{Z}}$ , we can consider the image of  $\{f^k(x)\}_{k \in \mathbb{Z}}$  under  $\psi_\gamma$  in  $\mathring{D}$ . The image is a set of ten points, which we label from left to right by  $z_0, z_1, \dots, z_9$ . For each  $0 \leq i \leq 8$ , we draw an oriented, straight-line arc  $p_i$  from  $z_i$  to  $z_{i+1}$ . Let  $\hat{z}_i$  and  $\hat{p}_i$  denote the lifts of  $z_i$  and  $p_i$ , respectively, to  $S^2$ . Let  $[\hat{s}_i] \in \text{MOD}^+(S_{0,10})$  denote the clockwise half-twist around  $\hat{p}_i$  (coming from the ambient standard orientation in  $\mathbb{R}^3$ ), and recall that the half-twists  $[\hat{s}_i^\pm]$  generate  $\text{MOD}^+(S_{0,10})$ . Our goal is to realize  $[f^2] \in \text{MOD}^+(S_{0,10})$  as a product of the  $[\hat{s}_i^\pm]$ 's.

Let  $[s_i] \in \text{MOD}^+(\mathring{D}_{10})$  denote the counter-clockwise half-twist about  $p_i$ , and notice that  $[\psi_\gamma^{-1} \circ s_i \circ \psi_\gamma] = [\hat{s}_i]$  once we extend  $\psi_\gamma^{-1} \circ s_i \circ \psi_\gamma$  to  $S_{0,10}$  by fixing  $\gamma$ . Since  $f$  doesn't fix  $\gamma$  (although  $\gamma$  does have order 2),  $f$  doesn't directly descend to a homeomorphism of  $\mathring{D}_{10}$ . However, there exists  $f' \in [f] \in \text{MOD}^\pm(S_{0,10})$  such that  $f'(\gamma) = \gamma$  and  $f'(\hat{p}_i) = f(\hat{p}_i)$  for all  $i$ . Indeed, this follows from the fact that the union  $\cup_i \hat{p}_i$  does not separate  $S^2$  and does not contain  $\gamma$  or  $f^{-1}(\gamma)$ . We use  $f'$  to denote the induced homeomorphism of  $\mathring{D}_{10} \cong S_{0,11}$ . Then

$$f'(p_i) = \psi_\gamma \circ f(\hat{p}_i).$$

In K3entropy, `arcs.c`, we plot each arc  $f'(p_i)$  in  $\mathring{D}_{10}$  and record its so-called *arc-data*, which we describe later. See the README file in GitHub for a description of that code.

In what follows, we use  $s_i \cdot s_k$  to denote the composition  $s_i \circ s_k$ <sup>1</sup>. From the arc-data produced by `arcs.c`, the algorithm described in Section 3.6.1 finds  $i_k \in \{0, 1, \dots, 8\}$  and  $n_k \in \{1, -1\}$  such that

$$g := s_{i_0}^{n_0} \cdot s_{i_1}^{n_1} \cdot \dots \cdot s_{i_\ell}^{n_\ell} \in \text{Homeo}^+(\mathring{D}_{10})$$

maps each  $f'(p_i)$  to  $p_i$  up to homotopy in  $\mathring{D}_{10}$ . This algorithm is implemented in K3entropy, `mclass.c`. Then

$$\hat{g} := \hat{s}_{i_0}^{n_0} \cdot \hat{s}_{i_1}^{n_1} \cdot \dots \cdot \hat{s}_{i_\ell}^{n_\ell} \in \text{Homeo}^+(S_{0,10} \setminus \{\gamma\})$$

maps each  $f(\hat{p}_i)$  to  $\hat{p}_i$  up to homotopy in  $S_{0,11} = S_{0,10} \setminus \{\gamma\}$ . Thus  $[\hat{g} \circ f] \in \text{MOD}^\pm(S_{0,10})$  is an orientation-reversing mapping class which fixes each  $\hat{p}_i$ ; there is only one such mapping class. Indeed, let  $\nu$  be the unique (up to homotopy in  $S_{0,10}$ ) arc from  $\hat{z}_8$  to  $\hat{z}_0$  that's disjoint from each  $\hat{p}_i$ , and let  $L$  be the closed loop defined by the concatenation  $\hat{p}_0 \cdot \hat{p}_1 \cdot \dots \cdot \hat{p}_8 \cdot \nu$ . Let  $R \in \text{Homeo}(S_{0,10})$  be a reflection through  $L$ . Notice  $\hat{g} \circ f \circ R$  is orientation-preserving and fixes each  $p_i$ , so it's the trivial mapping class. Thus  $[\hat{g} \circ f] = [R]$ .

We are not quite done, since we wanted to realize  $[f^2] \in \text{MOD}^+(S_{0,10})$  as a product of the  $[\hat{s}_i^\pm]$ 's. Notice that  $[R \circ \hat{s}_i^\pm] = [\hat{s}_i^\mp \circ R]$  for all  $i$ . Therefore

$$\begin{aligned} f^2 &\cong \hat{g}^{-1} \circ R \circ \hat{g}^{-1} \circ R \\ &\cong (\hat{s}_{i_\ell}^{-n_\ell} \cdot \hat{s}_{i_{\ell-1}}^{-n_{\ell-1}} \cdot \dots \cdot \hat{s}_{i_0}^{-n_0}) \circ (\hat{s}_{i_\ell}^{n_\ell} \cdot \hat{s}_{i_{\ell-1}}^{n_{\ell-1}} \cdot \dots \cdot \hat{s}_{i_0}^{n_0}) \\ (3.13) \quad &\cong \hat{g}^{-1} \circ \bar{g} \end{aligned}$$

where  $\bar{g}$  reverses the order of the product decomposition of  $\hat{g}$ , meaning

$$\bar{g} := \hat{s}_{i_\ell}^{n_\ell} \cdot \hat{s}_{i_{\ell-1}}^{n_{\ell-1}} \cdot \dots \cdot \hat{s}_{i_0}^{n_0}.$$

**3.6.1. An algorithm for computing  $g$ .** Let  $f', z_i, p_i$ , and  $s_i^\pm$  be as above, but consider the more general case of  $\mathring{D}_n$ . In what follows, we consider the  $z_i$  as marked points so that homeomorphisms of  $\mathring{D}_n$  permute the marked points.

Let  $\text{MOD}_0^+(\mathring{D}_n) \subset \text{MOD}^+(\mathring{D}_n)$  denote the subgroup generated by homotopy classes of homeomorphisms that fix the marked point  $z_0$ . The following lemma will be useful later:

**Lemma 3.14.** *The subgroup  $\text{MOD}_0^+(\mathring{D}_n)$  satisfies*

$$\text{MOD}_0^+(\mathring{D}_n) = \langle [\tau_{p_0}], [s_1], [s_2], \dots, [s_{n-2}] \rangle$$

where  $\tau_{p_0} = s_0 \cdot s_0$  is a full twist around  $p_0$ .

<sup>1</sup>This convention is consistent with the Flipper program.

*Proof of Lemma 3.14.* For each  $0 \leq i < j \leq n-1$ , define

$$A_{i,j} := (s_{j-1}.s_{j-2} \dots s_{i+1}).s_i^2.(s_{j-1}.s_{j-2} \dots s_{i+1})^{-1}.$$

Then  $A_{i,j}$  represents a Dehn twist around some loop containing just  $z_i$  and  $z_j$ , and notice that  $A_{0,1} = \tau_{p_0}$ . Moreover, the collection  $\{A_{i,j} \mid 0 \leq i < j \leq n-1\}$  generates the pure mapping class group  $\text{PMOD}^+(\mathring{D}_n)$  (see [12]). Therefore  $\text{PMOD}^+(\mathring{D}_n) \leq \langle [\tau_{p_0}], [s_1], [s_2], \dots, [s_{n-2}] \rangle$ . Finally, any  $[\varphi] \in \text{MOD}_0^+(\mathring{D}_n)$  can be written as a product of  $[\varphi_1] \in \text{PMOD}^+(\mathring{D}_n)$  and  $[\varphi_2] \in \langle [s_1], [s_2], \dots, [s_{n-2}] \rangle$ , proving the lemma.  $\square$

For all  $0 \leq i < n-1$ , we will find  $g_i \in \text{Homeo}^+(\mathring{D}_n)$  such that  $g_i(p_k) \cong p_k$  for each  $0 \leq k < i$  and  $g_i \circ g_{i-1} \circ \dots \circ g_0 \circ f(p_i) \cong p_i$ . Then  $g := g_{n-1} \circ g_{n-2} \circ \dots \circ g_0$  satisfies  $g \circ f'(p_i) \cong p_i$  for each  $0 \leq i < n-1$ , as desired. We will determine  $g_i$  inductively in the following way.

**Base case:** Choose a finite sequence half-twists  $t_1, \dots, t_\ell \in \{s_k^\pm \mid 0 \leq k < n-1\}$  such that

$$g_0 := t_\ell.t_{\ell-1} \dots t_1 \in \text{Homeo}^+(\mathring{D}_n)$$

satisfies  $g_0 \circ f'(p_0) \cong p_0$ . The half-twists  $t_k$  can be readily obtain from  $f'(p_0)$  by recording the sequence of marked points that  $f'(p_0)$  traverses, noting whether the arc passes over or under each marked point. We refer to this sequence of marked points and over/under information as the *arc-data* of  $f(p_0)$ . The process is straightforward and we omit details; one can follow along in Figures 3 and 4 to get a sense of it. The idea is to “drag” the endpoint of  $f'(p_0)$  backwards along the arc  $f'(p_0)$  towards its starting point using half-twists.

**Inductive case:** Let  $i > 0$ , and assume that  $g_k$  for  $0 \leq k < i$  have been constructed.

**Step 1.** Define  $P_{i-1} := p_0.p_1 \dots p_{i-1}$ . Define a homotopy  $C : [0, 1] \times \mathring{D} \rightarrow \mathring{D}$  such that:

- $C_0$  is the identity;
- $C_1(P_{i-1})$  is a point;
- for all  $t \in [0, 1)$ ,  $C_t$  is a homeomorphism satisfying  $C_t(P_{i-1}) \subset P_{i-1}$ ;
- for all  $t \in [0, 1]$ ,  $C_t$  fixes  $z_k$  whenever  $k > i$ .

Let  $c := C_1$ . Considering  $c(P_{i-1})$  to be a marked point along with the  $z_k$ 's for  $i < k < n-1$ ,  $c$  defines a continuous map  $c : \mathring{D}_n \rightarrow \mathring{D}_{n-i}$ . We take generators  $\{s_k \mid i \leq k < n-1\}$  for  $\mathring{D}_{n-i}$ , where  $s_k$  is a counter-clockwise half-twist about  $c(p_k)$ . Notice that, for  $i < k < n-1$ , one has

$$(3.15) \quad s_k \circ c \cong c \circ s_k.$$

Moreover,  $s_i$  admits the following lift by  $c$ :

$$(3.16) \quad s_i \circ c \cong c \circ (s_i.s_{i-1} \dots s_1.s_0.s_0.s_1 \dots s_{i-1}.s_i).$$

**Step 2.** Consider the arcs  $\alpha_i := g_{i-1} \circ g_{i-2} \circ \dots \circ g_0 \circ f'(p_i) \subset \mathring{D}_n$  and  $c(\alpha_i) \subset \mathring{D}_{n-i}$ . Choose  $\tilde{g}_i \in \text{Homeo}^+(\mathring{D}_{n-i})$  such that  $\tilde{g}_i(c(\alpha_i)) \cong c(p_i)$  in  $\mathring{D}_{n-i}$ . Notice that  $c(\alpha_i)$  and  $c(p_i)$  both have starting point  $c(P_{i-1})$ . Therefore,  $[\tilde{g}_i] \in \text{MOD}_0^+(\mathring{D}_{n-i})$ , and by Lemma 3.14,  $[\tilde{g}_i] \in \langle [\tau_{c(p_i)}], [s_{i+1}], [s_{i+2}], \dots, [s_{n-2}] \rangle$ . Just like in the base case, a presentation for  $[\tilde{g}_i]$  in terms of those generators can be obtained from the arc-data of  $c(\alpha_i)$ .

**Step 3.** By applying 3.15 and 3.16, we can lift  $\tilde{g}_i$  to  $g'_i \in \text{Homeo}^+(\mathring{D}_n)$  satisfying  $\tilde{h} \circ c = c \circ \hat{h}$ . The homeomorphism  $g'_i$  fixes  $p_k$  whenever  $0 \leq k < i$ . Moreover,  $g'_i$  maps  $\alpha_i$  to  $p_i$  up to twisting around  $P_{i-1}$ ; we make this precise in the next step.

**Step 4.** First, we show that there exists some  $h \in \text{Homeo}(\mathring{D}_n)$  such that  $h(p_k) \cong p_k$  for each  $1 \leq k < i$  and  $g'_i(\alpha_i) \cong h(p_i)$ . Indeed, notice that deleting open neighborhoods of either  $P_{i-1}.\alpha_i$  or  $P_i := P_{i-1}.p_i$  results in a surface with  $n-i-2$  marked points and one boundary component. Then the existence of  $h$  follows from the classification of finite-type surfaces.

Now, let  $\tau_{P_{i-1}}, \tau_{P_i}$  denote the counter-clockwise Dehn twists about  $P_{i-1}$  and  $P_i$ , respectively.

**Claim 3.17.**  $[h] \in \langle [\tau_{P_{i-1}}], [\tau_{P_i}], [s_{i+1}], [s_{i+2}], \dots, [s_{n-2}] \rangle$

*Proof of Claim 3.17.* Let  $\gamma$  be a simple closed curve in  $\mathring{D}_n$  (not containing marked points) isotopic to  $P_{i-1}$ . Then  $h(\gamma) \cong \gamma$ . The curve  $\gamma$  separates  $\mathring{D}_n$  into two  $[h]$ -invariant components, which we denote by  $A$  and  $B$ . Therefore, there exists a decomposition  $[g'_i] = [f_1 \circ f_2]$  with  $f_{1,2} \in \text{Homeo}(\mathring{D}_n)$  satisfying  $f_1|_B = id$  and  $f_2|_A = id$ . One of these components, say  $A$ , is itself an open disk with marked points  $z_0, z_1, \dots, z_i$ . Since  $h(p_k) \cong p_k$  for each  $1 \leq k < i$ ,  $[h]$  restricts to the trivial mapping class on  $A$ . So we can take  $f_1 = id$ . Since  $p_i$  and  $g'_i(\alpha_i)$  both end at  $z_{i+1}$  and  $h(p_i) = g'_i(\alpha_i)$ , we have  $h(z_{i+1}) = z_{i+1}$ . But  $h(z_{i+1}) = f_2(z_{i+1})$ ,

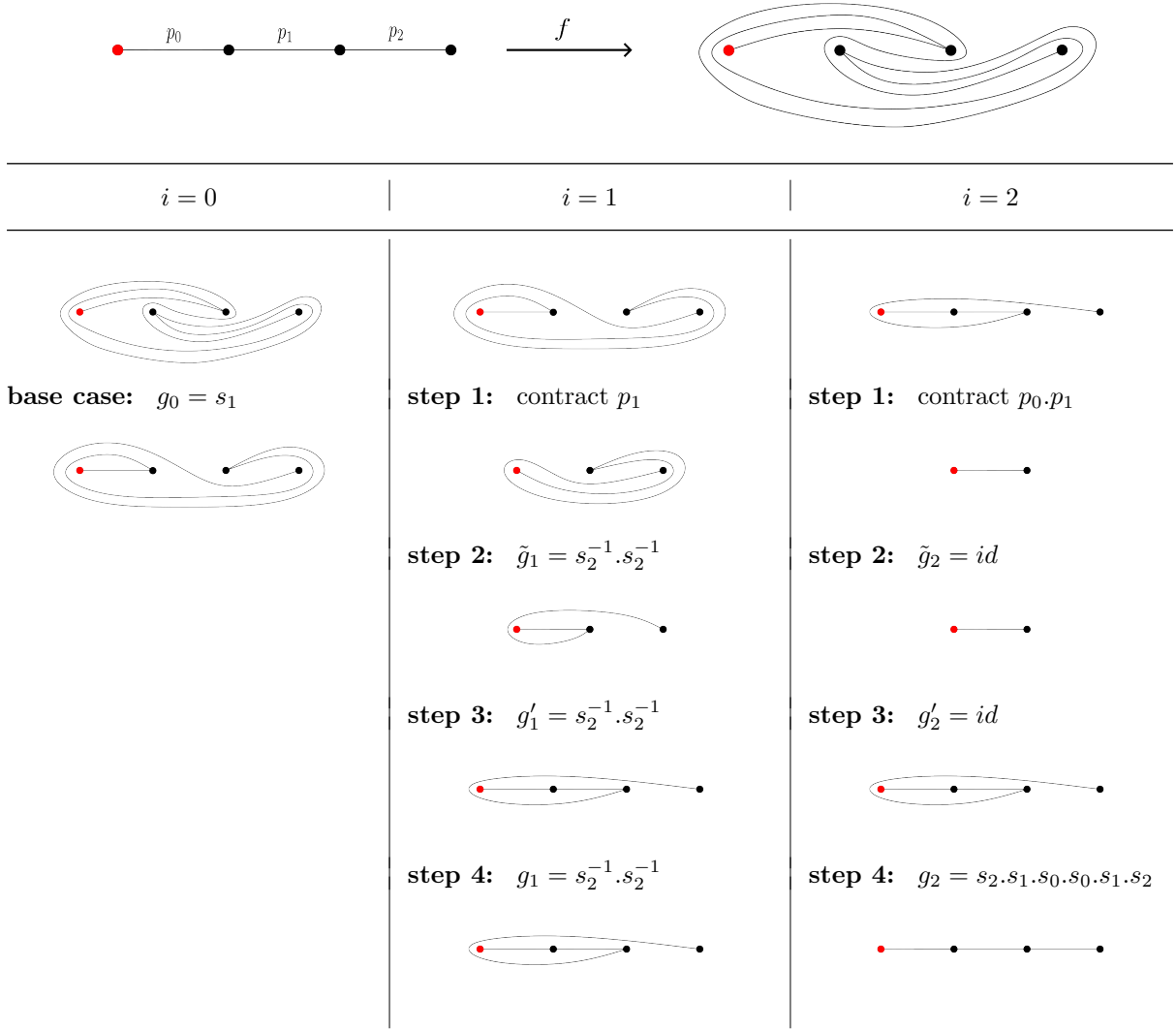


FIGURE 3. The algorithm described in Section 3.6.1 applied in a simple case. We obtain  $g = g_2 \circ g_1 \circ g_0 = s_2.s_1.s_0.s_0.s_1.s_2.s_2^{-1}.s_2^{-1}.s_1$ .

and, in light of Lemma 3.14, it's straightforward to see that  $[f_2] \in \langle [\tau_\gamma], [\tau_{P_i}], [s_{i+1}], [s_{i+2}], \dots, [s_{n-2}] \rangle$ , concluding the proof.  $\square$

Each of  $[\tau_{P_i}], [s_{i+1}], [s_{i+2}], \dots, [s_{n-2}]$  fixes  $p_i$ . Therefore, using Claim 3.17 and that  $h(p_i) = g'_i(\alpha_i)$ , we find that  $p_i = \tau_{P_{i-1}}^k(g'_i(\alpha_i))$  for some  $k \in \mathbb{Z}$ . One could let  $g_i := \tau_{P_{i-1}}^k \circ g'_i$ , but we instead set  $g_i := \tau_{P_i}^{-k} \circ \tau_{P_{i-1}}^k \circ g'_i$ . Notice that  $\tau_{P_i}$  and  $\tau_{P_{i-1}}$  commute; this can be seen by realizing  $\tau_{P_i}$  and  $\tau_{P_{i-1}}$  as twists around disjoint closed loops. So

$$\begin{aligned} \tau_{P_i}^{-k} \tau_{P_{i-1}}^k &= (\tau_{P_i}^{-1} \tau_{P_{i-1}})^k \\ &= ((s_{i-1}^{-1}.s_{i-2}^{-1} \dots s_0^{-1}).(s_0^{-1}.s_1^{-1} \dots s_{i-1}^{-1}))^k \end{aligned}$$

and we have

$$g_i = ((s_{i-1}^{-1}.s_{i-2}^{-1} \dots s_0^{-1}).(s_0^{-1}.s_1^{-1} \dots s_{i-1}^{-1}))^k \circ g'_i,$$

concluding the inductive step.

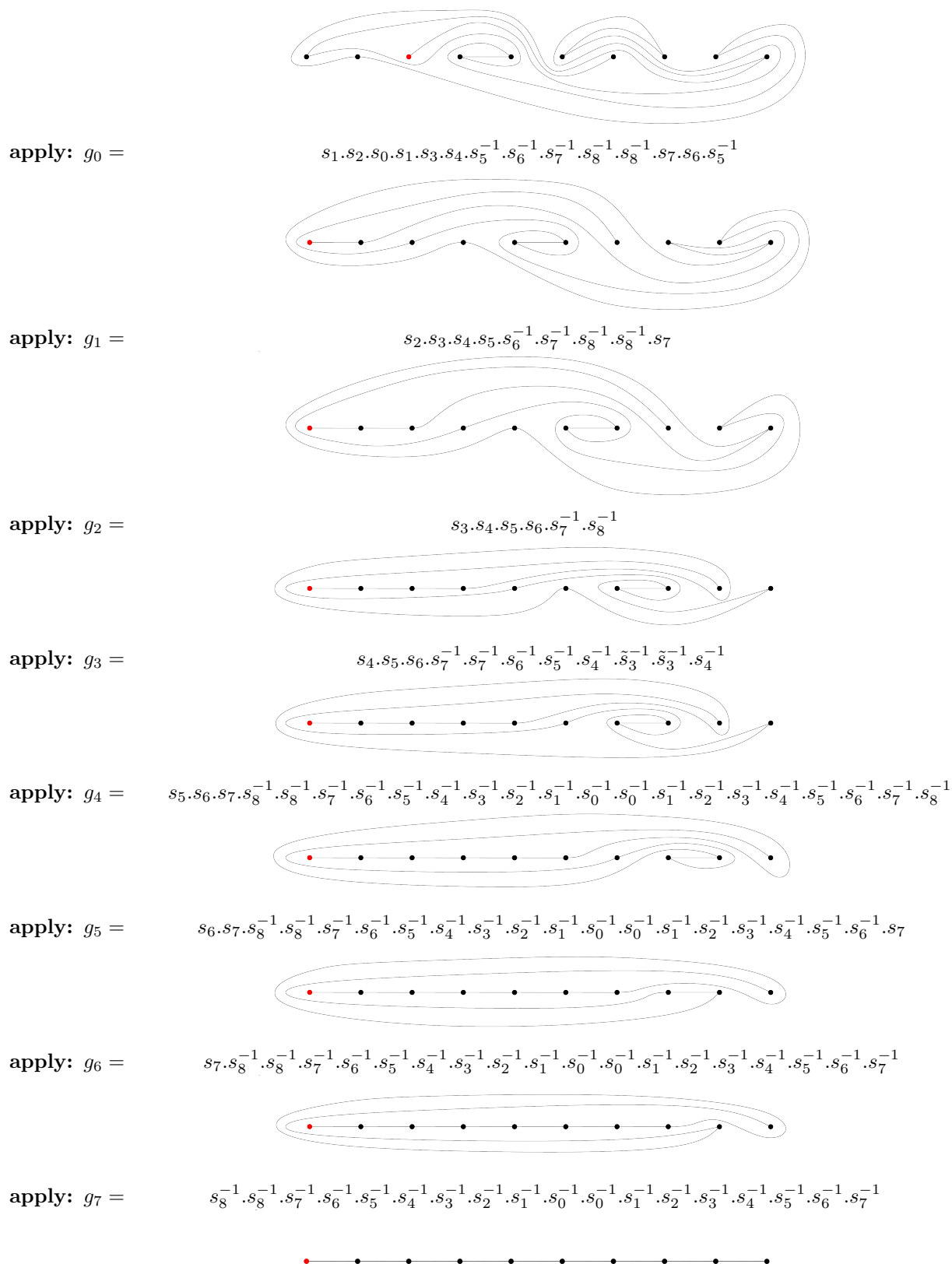


FIGURE 4. The algorithm described in Section 3.6.1, applied to the 10-periodic point approximated in Section 3.4. See Section 3.6.1 for notation.

3.6.2. *Application to our case and using Flipper.* In Figure 4, we apply the algorithm described in Section 3.6.1 to compute  $g$ . We find:

$$\begin{aligned}
g &= g_7 \circ g_6 \circ \cdots \circ g_0 \\
&= (s_8^{-1} \cdot s_8^{-1} \cdot s_7^{-1} \cdot s_6^{-1} \cdot s_5^{-1} \cdot s_4^{-1} \cdot s_3^{-1} \cdot s_2^{-1} \cdot s_1^{-1} \cdot s_0^{-1} \cdot s_0^{-1} \cdot s_1^{-1} \cdot s_2^{-1} \cdot s_3^{-1} \cdot s_4^{-1} \cdot s_5^{-1} \cdot s_6^{-1} \cdot s_7^{-1}) \\
&\quad \circ (s_7 \cdot s_8^{-1} \cdot s_8^{-1} \cdot s_7^{-1} \cdot s_6^{-1} \cdot s_5^{-1} \cdot s_4^{-1} \cdot s_3^{-1} \cdot s_2^{-1} \cdot s_1^{-1} \cdot s_0^{-1} \cdot s_0^{-1} \cdot s_1^{-1} \cdot s_2^{-1} \cdot s_3^{-1} \cdot s_4^{-1} \cdot s_5^{-1} \cdot s_6^{-1} \cdot s_7^{-1}) \\
&\quad \circ (s_6 \cdot s_7 \cdot s_8^{-1} \cdot s_8^{-1} \cdot s_7^{-1} \cdot s_6^{-1} \cdot s_5^{-1} \cdot s_4^{-1} \cdot s_3^{-1} \cdot s_2^{-1} \cdot s_1^{-1} \cdot s_0^{-1} \cdot s_0^{-1} \cdot s_1^{-1} \cdot s_2^{-1} \cdot s_3^{-1} \cdot s_4^{-1} \cdot s_5^{-1} \cdot s_6^{-1} \cdot s_7) \\
&\quad \circ (s_5 \cdot s_6 \cdot s_7 \cdot s_8^{-1} \cdot s_8^{-1} \cdot s_7^{-1} \cdot s_6^{-1} \cdot s_5^{-1} \cdot s_4^{-1} \cdot s_3^{-1} \cdot s_2^{-1} \cdot s_1^{-1} \cdot s_0^{-1} \cdot s_0^{-1} \cdot s_1^{-1} \cdot s_2^{-1} \cdot s_3^{-1} \cdot s_4^{-1} \cdot s_5^{-1} \cdot s_6^{-1} \cdot s_7^{-1} \cdot s_8^{-1}) \\
&\quad \circ (s_4 \cdot s_5 \cdot s_6 \cdot s_7^{-1} \cdot s_7^{-1} \cdot s_6^{-1} \cdot s_5^{-1} \cdot s_4^{-1} \cdot s_3^{-1} \cdot s_2^{-1} \cdot s_1^{-1} \cdot s_0^{-1} \cdot s_0^{-1} \cdot s_1^{-1} \cdot s_2^{-1} \cdot s_3^{-1} \cdot s_4^{-1}) \circ (s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7^{-1} \cdot s_8^{-1}) \\
&\quad \circ (s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6^{-1} \cdot s_7^{-1} \cdot s_8^{-1} \cdot s_8^{-1} \cdot s_7) \circ (s_1 \cdot s_2 \cdot s_0 \cdot s_1 \cdot s_3 \cdot s_4 \cdot s_5^{-1} \cdot s_6^{-1} \cdot s_7^{-1} \cdot s_8^{-1} \cdot s_8^{-1} \cdot s_7 \cdot s_6 \cdot s_5^{-1}).
\end{aligned}$$

As we saw in 3.13,  $[f^2]$  is readily obtained from  $g$ . For completeness, we include its mapping class below:

$$\begin{aligned}
f^2 &\cong [(\hat{s}_5 \cdot \hat{s}_6^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_8 \cdot \hat{s}_8 \cdot \hat{s}_7 \cdot \hat{s}_6 \cdot \hat{s}_5 \cdot \hat{s}_4^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_0^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_1^{-1}) \circ (\hat{s}_7^{-1} \cdot \hat{s}_8 \cdot \hat{s}_8 \cdot \hat{s}_7 \cdot \hat{s}_6 \cdot \hat{s}_5^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_2^{-1}) \\
&\quad \circ (\hat{s}_8 \cdot \hat{s}_7 \cdot \hat{s}_6^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_3^{-1}) \circ (\hat{s}_4 \cdot \hat{s}_3 \cdot \hat{s}_2 \cdot \hat{s}_1 \cdot \hat{s}_0 \cdot \hat{s}_1 \cdot \hat{s}_2 \cdot \hat{s}_3 \cdot \hat{s}_4 \cdot \hat{s}_5 \cdot \hat{s}_6 \cdot \hat{s}_7 \cdot \hat{s}_8 \cdot \hat{s}_7^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_5^{-1}) \\
&\quad \circ (\hat{s}_8 \cdot \hat{s}_7 \cdot \hat{s}_6 \cdot \hat{s}_5 \cdot \hat{s}_4 \cdot \hat{s}_3 \cdot \hat{s}_2 \cdot \hat{s}_1 \cdot \hat{s}_0 \cdot \hat{s}_1 \cdot \hat{s}_2 \cdot \hat{s}_3 \cdot \hat{s}_4 \cdot \hat{s}_5 \cdot \hat{s}_6 \cdot \hat{s}_7 \cdot \hat{s}_8 \cdot \hat{s}_8 \cdot \hat{s}_7^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_5^{-1}) \\
&\quad \circ (\hat{s}_7^{-1} \cdot \hat{s}_6 \cdot \hat{s}_5 \cdot \hat{s}_4 \cdot \hat{s}_3 \cdot \hat{s}_2 \cdot \hat{s}_1 \cdot \hat{s}_0 \cdot \hat{s}_1 \cdot \hat{s}_2 \cdot \hat{s}_3 \cdot \hat{s}_4 \cdot \hat{s}_5 \cdot \hat{s}_6 \cdot \hat{s}_7 \cdot \hat{s}_8 \cdot \hat{s}_8 \cdot \hat{s}_7^{-1} \cdot \hat{s}_6^{-1}) \\
&\quad \circ (\hat{s}_7 \cdot \hat{s}_6 \cdot \hat{s}_5 \cdot \hat{s}_4 \cdot \hat{s}_3 \cdot \hat{s}_2 \cdot \hat{s}_1 \cdot \hat{s}_0 \cdot \hat{s}_1 \cdot \hat{s}_2 \cdot \hat{s}_3 \cdot \hat{s}_4 \cdot \hat{s}_5 \cdot \hat{s}_6 \cdot \hat{s}_7 \cdot \hat{s}_8 \cdot \hat{s}_8 \cdot \hat{s}_7^{-1}) \\
&\quad \circ (\hat{s}_7 \cdot \hat{s}_6 \cdot \hat{s}_5 \cdot \hat{s}_4 \cdot \hat{s}_3 \cdot \hat{s}_2 \cdot \hat{s}_1 \cdot \hat{s}_0 \cdot \hat{s}_1 \cdot \hat{s}_2 \cdot \hat{s}_3 \cdot \hat{s}_4 \cdot \hat{s}_5 \cdot \hat{s}_6 \cdot \hat{s}_7 \cdot \hat{s}_8 \cdot \hat{s}_8)] \\
&\quad \circ [(\hat{s}_5^{-1} \cdot \hat{s}_6 \cdot \hat{s}_7 \cdot \hat{s}_8^{-1} \cdot \hat{s}_8^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_4 \cdot \hat{s}_3 \cdot \hat{s}_1 \cdot \hat{s}_0 \cdot \hat{s}_2 \cdot \hat{s}_1) \circ (\hat{s}_7 \cdot \hat{s}_8^{-1} \cdot \hat{s}_8^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_5 \cdot \hat{s}_4 \cdot \hat{s}_3 \cdot \hat{s}_2) \\
&\quad \circ (\hat{s}_8^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_6 \cdot \hat{s}_5 \cdot \hat{s}_4 \cdot \hat{s}_3) \circ (\hat{s}_4^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_0^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_8^{-1} \cdot \hat{s}_7 \cdot \hat{s}_6 \cdot \hat{s}_5) \\
&\quad \circ (\hat{s}_8^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_0^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_8^{-1} \cdot \hat{s}_7 \cdot \hat{s}_6 \cdot \hat{s}_5) \\
&\quad \circ (\hat{s}_7 \cdot \hat{s}_6^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_0^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_8^{-1} \cdot \hat{s}_8^{-1} \cdot \hat{s}_7 \cdot \hat{s}_6) \\
&\quad \circ (\hat{s}_7^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_0^{-1} \cdot \hat{s}_1^{-1} \cdot \hat{s}_2^{-1} \cdot \hat{s}_3^{-1} \cdot \hat{s}_4^{-1} \cdot \hat{s}_5^{-1} \cdot \hat{s}_6^{-1} \cdot \hat{s}_7^{-1} \cdot \hat{s}_8^{-1} \cdot \hat{s}_8^{-1})].
\end{aligned}$$

In K3entropy, ‘FlipperDilatation.ipynb’<sup>2</sup>, we use Flipper to find the stretch factor of  $[f^2]$ , and obtain

$$\lambda([f^2]) \approx 8.1998.$$

It follows that

$$h_{top}(f, X(\mathbb{R})) \gtrsim \frac{1}{2} \ln 8.1998,$$

and by 2.5,

$$\frac{h_{top}(f, X(\mathbb{R}))}{h_{top}(f, X(\mathbb{C}))} \gtrsim 0.58.$$

## Appendices

### A. PROOF OF THE SHADOWING LEMMA

*Proof of Lemma 3.2.* To start, we set up a fixed point problem. Let  $\varepsilon := 12C\delta$  so that  $B_\varepsilon(0) \subset U_i$ . Pick a smooth bump function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  vanishing outside of  $B_\varepsilon(0)$  and satisfying  $g|_{B_{\frac{\varepsilon}{2}}(0)} \equiv 1$ . Furthermore, choose  $g$  such that  $\|Dg\|_{C^0(\mathbb{R}^2)} < \frac{3}{\varepsilon}$ . For each  $1 \leq i \leq n$ , define  $H_i, L_i, M_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$\begin{aligned}
L_i &:= (Dh_i^c)_0, \\
H_i &:= gh_i^c + (1-g)L_i, \\
M_i &:= h_i - L_i.
\end{aligned}$$

Furthermore, define  $H, L, M : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  by

$$\begin{aligned}
H(w_0, \dots, w_{n-1}) &:= (H_{n-1}(w_{n-1}), H_0(w_0), \dots, H_{n-2}(w_{n-2})), \\
L(w_0, \dots, w_{n-1}) &:= (L_{n-1}(w_{n-1}), L_0(w_0), \dots, L_{n-2}(w_{n-2})), \\
M(w_0, \dots, w_{n-1}) &:= (M_{n-1}(w_{n-1}), M_0(w_0), \dots, M_{n-2}(w_{n-2})).
\end{aligned}$$

<sup>2</sup>Flipper uses right Dehn twists. The distinction between left and right is not important to us, since they’re the same up to conjugation by a homeomorphism, which preserves topological entropy.

We hope to find a fixed point  $z = (z_0, \dots, z_{n-1})$  for  $H$  in  $B_{\frac{\varepsilon}{2}}(\vec{0})$ . Here we are using the supremum norm  $\|\cdot\|_0$  on  $(\mathbb{R}^2)^n$  (see 1.4). Notice that we require  $z_0 \in B_{\frac{\varepsilon}{2}}(\vec{0})$  in order to lift  $z$  to an  $h$ -periodic point. Indeed, since  $H_i|_{B_{\frac{\varepsilon}{2}}(0)} \equiv h_i^c$ , one has  $H(z) = (h_{n-1}^c(z_{n-1}), h_0^c(z_0), \dots, h_{n-2}^c(z_{n-2}))$ . Thus  $(\tilde{x}_i) := (\varphi_i(z_i))$  is a fixed point for  $(w_0, \dots, w_{n-1}) \mapsto (h(w_{n-1}), h(w_0), \dots, h(w_{n-2}))$  as a function  $M^n \rightarrow M^n$ . Note that

$$x = H(x) = L(x) + M(x)$$

exactly when

$$(L - id)x = -M(x)$$

or in other words

$$x = -(L - id)^{-1}M(x).$$

So, we hope to find a fixed point for  $\mathcal{F} := -(L - id)^{-1}M$  in  $B_{\frac{\varepsilon}{2}}(\vec{0})$ .

**Claim A.1.**  $\mathcal{F}$  is a  $\frac{3}{4}$ -contraction mapping for  $\|\cdot\|_0$  whenever

$$\varepsilon < \frac{1}{16C\|D^2h_i^c\|_{C^0(B_\varepsilon(0)),\infty}}$$

for all  $i$ .

We must prove  $\sup_{w \in \mathbb{R}^{2n}} \|D((L - id)^{-1}M)_w\|_{op,0} \leq \frac{3}{4}$ . We treat  $\sup_{w \in \mathbb{R}^{2n}} \|D(L - id)^{-1}\|_{op,0}$  and  $\sup_{w \in \mathbb{R}^{2n}} \|DM\|_{op,0}$  separately. Now  $L$  is linear, so  $D(L - id)_w^{-1} = (L - id)^{-1}$  for all  $w$ . Thus  $\sup_{w \in \mathbb{R}^{2n}} \|D(L - id)^{-1}\|_{op,0} = C$ .

**Claim A.2.**  $\sup_{w \in \mathbb{R}^2} \|(DM)_w\|_{op,0} \leq \max_i \left( 8\varepsilon\|D^2h_i^c\|_{C^0(B_\varepsilon(0)),\infty} + 3\frac{\delta}{\varepsilon} \right)$

Assuming A.2, we have

$$\begin{aligned} \sup_{w \in \mathbb{R}^{2n}} \|D((L - id)^{-1}M)_w\|_{op,0} &\leq C \cdot \left( 8 \cdot \frac{1}{16C} + \frac{3}{12C} \right) \\ &= \frac{3}{4}, \end{aligned}$$

proving A.1. To begin the proof of A.2, define  $N_i^c : B_\varepsilon(0) \rightarrow \mathbb{R}^2$  by

$$N_i^c := h_i^c - L_i.$$

The function  $N_i^c$  is essentially the non-linear part of  $h_i^c$  at 0, with the caveat that  $N_i^c$  might not fix the origin. Notice  $M_i = H_i - L_i = g_i(h_i^c - L_i) = g_i N_i^c$  implies

$$\begin{aligned} \sup_{w \in \mathbb{R}^2} \|(DM_i)_w\|_{op} &\leq \|DM_i\|_{C^0(\mathbb{R}^2)} \\ &\leq \|Dg\|_{C^0(\mathbb{R}^2)} \|N_i^c\|_{C^0(B_\varepsilon(0))} + \|g_i\|_{C^0(\mathbb{R}^2)} \|DN_i^c\|_{C^0(B_\varepsilon(0))} \\ &\leq \frac{3}{\varepsilon} \|N_i^c\|_{C^0(B_\varepsilon(0))} + \|DN_i^c\|_{C^0(B_\varepsilon(0))}. \end{aligned}$$

Thus

$$\begin{aligned} \sup_w \|(DM)_w\|_{op,0} &= \sup_w \max_i (\|(DM_i)_w\|_{op}) \\ &\leq \max_i \left( \frac{3}{\varepsilon} \|N_i^c\|_{C^0(B_\varepsilon(0))} + \|DN_i^c\|_{C^0(B_\varepsilon(0))} \right). \end{aligned}$$

Thus A.2 will follow from the following two subclaims.

**Claim A.3.**  $\|N_i^c\|_{C^0(B_\varepsilon(0))} \leq 2\varepsilon^2\|D^2h_i^c\|_{C^0(B_\varepsilon(0)),\infty} + \delta$

Well, for all  $v \in B_\varepsilon(0)$ , Taylor's theorem implies that there exist some  $z_1, z_2 \in B_\varepsilon(0)$  such that

$$N_i^c(v) - h_i^c(0) = \frac{1}{2} \begin{pmatrix} v \\ v \end{pmatrix}^t \cdot \left( (D^2(h_i^c)_1)_{z_1} \quad (D^2(h_i^c)_2)_{z_2} \right) \cdot \begin{pmatrix} v \\ v \end{pmatrix}$$

where  $h_i^c = ((h_i^c)_1, (h_i^c)_2)$ . Thus

$$\|N_i^c(v)\|_2 \leq |v|^2 \max_{k=1,2} \|(D^2(h_i^c)_k)_{z_k}\|_F + \delta.$$

Since

$$\max_{k=1,2} \|(D^2(h_i^c)_k)_{z_k}\|_F \leq 2\|D^2h_i^c\|_{C^0(B_\varepsilon(0)),\infty}$$

and  $|v| < \varepsilon$ , we obtain

$$\|N_i^c\|_{C^0(B_\varepsilon(0))} \leq 2\varepsilon^2\|D^2h_i^c\|_{C^0(B_\varepsilon(0)),\infty} + \delta.$$

**Claim A.4.**  $\|DN_i^c\|_{C^0(B_\varepsilon(0))} \leq 2\varepsilon\|D^2h_i^c\|_{C^0(B_\varepsilon(0)),\infty}$

Notice

$$(DN_i^c)_w = (Dh_i^c)_w - (Dh_i^c)_0.$$

Therefore,

$$\begin{aligned} \|(DN_i^c)_w\|_{op} &= \|(Dh_i^c)_w - (Dh_i^c)_0\|_{op} \\ &\leq \|(Dh_i^c)_w - (Dh_i^c)_{x_i}\|_F \\ &\leq 2\varepsilon\|D^2h_i^c\|_{C^0(B_\varepsilon(0)),\infty}. \end{aligned}$$

Claim A.1 is proven, but we are not quite done, since we need  $z \in B_{\frac{\varepsilon}{2}}(\vec{0})$ . Well,

$$z = \lim_i(\mathcal{F}^i(\hat{z}))$$

for any  $\hat{z} \in \mathbb{R}^{2n}$ . Therefore,

$$\begin{aligned} \|z - \hat{z}\|_0 &\leq \sum_i \|(-\mathcal{F}^{i+1}(\hat{z}) - \mathcal{F}^i(\hat{z}))\|_0 \\ &\leq \sum_i \left(\frac{3}{4}\right)^i \|\mathcal{F}(\hat{z}) - \hat{z}\|_0 \\ &= 4\|\mathcal{F}(\hat{z}) - \hat{z}\|_0. \end{aligned}$$

In particular, substituting  $\vec{0}$  for  $\hat{z}$ ,

$$\begin{aligned} \|z\|_0 &\leq 4\|\mathcal{F}(0)\|_0 \\ &\leq 4C\| -M(0) - (L - id)0\|_0 \\ &= 4C\|M(0)\|_0 \\ &= 4C\|H(0) + L(0)\|_0 \\ &= 4C\|H(0)\|_0 \leq C\delta. \end{aligned}$$

We may conclude, since  $\delta = \frac{1}{12C}\varepsilon$  by assumption.  $\square$

## B. MISCELLANEOUS

**Lemma B.1.**  $X_A(\mathbb{R})$  is completely contained in the affine part of  $(\mathbb{P}^1)^3$ , and  $\{q < 0\} \subset \mathbb{R}^3$  is star-shaped about the origin.

*Proof.* In homogeneous coordinates, we find

$$\tilde{q}_A([0 : 1], [y_0 : y], [z_0 : z]) = (y_0^2 + y^2)(z_0^2 + z^2)$$

so, by the symmetry of  $\tilde{q}_A$ ,  $X_A(\mathbb{R})$  contains no points at infinity.

Now  $q(\vec{0}) = -2$ , so  $\vec{0} \in \{q < 0\}$ . If  $\{q < 0\}$  were not star-shaped about  $\vec{0}$ , the mean value theorem would imply that there exists  $p \in X(\mathbb{R})$  with  $p \in T_pX(\mathbb{R})$ . In other words,  $0 = q(p) = Dq_p(p)$ . Letting  $p = (x, y, z)$ , this amounts to

$$\begin{aligned} 0 &= (1 + x^2)(1 + y^2)(1 + z^2) + 10xyz - 2 = z(10xy + 2(1 + x^2)(1 + y^2)z) \\ &\quad + y(10xz + 2(1 + x^2)y(1 + z^2)) \\ &\quad + x(10yz + 2x(1 + y^2)(1 + z^2)). \end{aligned}$$

The resultant of the above two polynomials with respect to  $z$  is

$$\begin{aligned} R(x, y) = & 4 + 16x^2 + 8x^4 - 16x^6 + 4x^8 + 16y^2 - 252x^2y^2 + 400x^4y^2 \\ & - 148x^6y^2 + 16x^8y^2 + 8y^4 + 400x^2y^4 + 552x^4y^4 - 232x^6y^4 \\ & + 24x^8y^4 - 16y^6 - 148x^2y^6 - 232x^4y^6 + 316x^6y^6 + 16x^8y^6 \\ & + 4y^8 + 16x^2y^8 + 24x^4y^8 + 16x^6y^8 + 4x^8y^8. \end{aligned}$$

One can check with Mathematica that  $R$  admits exactly four real solutions:

$$(0, \pm\sqrt{1+\sqrt{2}}, (\pm\sqrt{1+\sqrt{2}}, 0),$$

none of which lift to  $X(\mathbb{R})$ . □

**Lemma B.2.** *We have*

$$\max_{(x,y,z) \in X(\mathbb{R})} |x| \approx 2.3.$$

*By symmetry, the same holds for the  $y$  and  $z$  coordinates.*

*Proof.* We omit the proof; it can be done by hand with lagrange multipliers. □

**Lemma B.3.** *Let  $S = \{(x, y) \in \mathbb{R}^2 \mid \mathcal{D}(x, y) > 0\}$ . The function  $\mathcal{D}$  satisfies the derivative bounds*

$$\max_{|\beta|=3} \sup_{(x,y) \in S} |D^\beta \mathcal{D}(x, y)| \leq 2000$$

*Proof.* We find

$$\partial_{xxx} \mathcal{D}(x, y) = -96x(1+y^2)^2$$

and

$$\partial_{xxy} \mathcal{D}(x, y) = 432y - 128x^2y(1+y^2) - 64(1+x^2)y(1+y^2).$$

If  $|x|, |y| \leq c$ , we calculate

$$|\partial_{xxx} \mathcal{D}(x, y)| \leq 96c(1+c^2)^2$$

and

$$|\partial_{xxy} \mathcal{D}(x, y)| = 432c + 128c^3(1+c^2) + 64c(1+c^2)^2.$$

The bound from Lemma B.2 and symmetry conclude the proof. □

**Lemma B.4.** *Considering  $f_A$  as a smooth function from an open neighborhood of  $X_A(\mathbb{R})$  into  $\mathbb{R}^3$ , we have*

$$\|D^2 f_A\|_{C^0(X(\mathbb{R})), \infty} \leq 9^2(2 \cdot \max(A, 1))^3$$

and

$$\|D f_A\|_{C^0(X(\mathbb{R})), \infty} \leq 3^2 \cdot \max(A, 1)^3.$$

*Proof.* Define

$$\alpha(x, y) := \frac{xy}{(1+x^2)(1+y^2)}.$$

In affine coordinates, one calculates

$$\sigma_1^A(x, y, z) = (-x - A\alpha(y, z), y, z),$$

$$\sigma_2^A(x, y, z) = (x, -y - A\alpha(x, z), z),$$

and

$$\sigma_3^A(x, y, z) = (x, y, -z - A\alpha(x, y)).$$

and notice that

$$\alpha(x, y) = \frac{1}{4} \left( \frac{1}{-x-i} + \frac{1}{-x+i} \right) \left( \frac{1}{-y-i} + \frac{1}{-y+i} \right).$$

It follows that, for  $x, y \in \mathbb{R}$ ,

$$|\alpha| \leq \frac{1}{4}(1+1)(1+1) = 1$$

$$|\partial_y(\alpha)|, |\partial_x(\alpha)| \leq \frac{1}{4}(1+1)(1+1) = 1$$

$$|\partial_{xy}(\alpha)| \leq \frac{1}{4}(1+1)(1+1) = 1$$

$$|\partial_{yy}(\alpha)|, |\partial_{xx}(\alpha)| \leq \frac{1}{4}2(1+1)(1+1) = 2$$

Then  $\|D\sigma_i^A\|_{X_A(\mathbb{R}),\infty} \leq \max(1, A)$ , and, by B.6,  $\|J^2(\sigma_i^A)\|_{X_A(\mathbb{R}),\infty} \leq 2 \cdot \max(A, 1)^2$ . The result follows from matrix multiplication and dimension counting.  $\square$

**Lemma B.5.** *If  $C_1, C_2 \in \mathbb{R}_{>0}$  and  $x, y \in \mathbb{R}$  are such that*

- $1 \leq \mathcal{D}(x, y) \leq C_1$  and
- $\|(DD)_{(x,y)}\|_\infty, \|(D^2\mathcal{D})_{(x,y)}\|_\infty \leq C_2$

then

- $\|(D^2p_\pm)_{(x,y)}\|_\infty \leq 10 + C_1^{0.5} + \frac{3}{4}C_2 + \frac{1}{8}C_2^2$  and
- $\|(Dp_\pm)_{(x,y)}\|_\infty \leq 5 + \frac{1}{2}C_1^{0.5} + \frac{1}{4}C_2$ .

*Proof.* With  $\alpha$  as in Lemma B.4 and  $\beta(x, y) := \frac{1}{(1+x^2)(1+y^2)}$ ,

$$p_\pm = -5\alpha \pm \frac{1}{2}\mathcal{D}^{\frac{1}{2}}\beta.$$

Notice that the same derivative bounds we established for  $\alpha$  in Lemma B.4 hold for  $\beta$ . Then

$$\begin{aligned} |\partial_j p_\pm| &\leq 5|\partial_j \alpha| + \frac{1}{2} \left( |\mathcal{D}^{0.5}| |\partial_j \beta| + |\beta| \left| \frac{\partial_j \mathcal{D}}{2\mathcal{D}^{0.5}} \right| \right) \\ &\leq 5 + \frac{1}{2}C_1^{0.5} + \frac{1}{4}C_2 \end{aligned}$$

and

$$\begin{aligned} |\partial_{ij} p_\pm| &\leq 5|\partial_{ij} \alpha| + \frac{1}{2} \left( \left| \frac{\partial_i \mathcal{D}}{2\mathcal{D}^{0.5}} \right| |\partial_j \beta| + |\mathcal{D}^{0.5}| |\partial_{ij} \beta| + |\beta| \left| \frac{\partial_j \mathcal{D} \partial_i \mathcal{D}}{4\mathcal{D}^{1.5}} \right| + |\beta| \left| \frac{\partial_{ij} \mathcal{D}}{2\mathcal{D}^{0.5}} \right| + |\partial_i \beta| \left| \frac{\partial_j \mathcal{D}}{2\mathcal{D}^{0.5}} \right| \right) \\ &\leq 10 + \frac{1}{2} \left( \frac{C_2}{2} + 2C_1^{0.5} + \frac{C_2^2}{4} + \frac{C_2}{2} + \frac{C_2}{2} \right) \\ &= 10 + C_1^{0.5} + \frac{3}{4}C_2 + \frac{1}{8}C_2^2, \end{aligned}$$

as desired.  $\square$

**B.1. Jets.** Consider  $C^\infty(\mathbb{R}^n, \mathbb{R})$ , the collection of smooth functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Let  $k \in \mathbb{N}$  and  $x_0 \in \mathbb{R}^n$ , and consider the equivalence relation on  $C^\infty(\mathbb{R}^n, \mathbb{R})$  defined by  $f \sim g$  if  $f(x_0) = g(x_0)$  and all partials of  $f$  and  $g$  at  $x_0$  of order  $\leq k$  agree. We call such an equivalence class a  $k$ -jet, and denote the set of classes by  $J_{x_0}^k(\mathbb{R}^n, \mathbb{R})$ . Furthermore, denote by  $J_{x_0}^k(\mathbb{R}^n, \mathbb{R})_0$  the collection of jets mapping  $x_0$  to 0. For each  $1 \leq i \leq n$ , let  $x_i \in C^\infty(\mathbb{R}^n, \mathbb{R})$  denote projection onto the  $i$ th coordinate. By Taylor's theorem,  $J_{x_0}^k(\mathbb{R}^n, \mathbb{R})_0$  is a linear space. Moreover, the collection of products

$$\mathcal{B}_{x_0} := \{(x_{i_1} - (x_0)_{i_1}) \cdots (x_{i_\ell} - (x_0)_{i_\ell}) \mid 1 \leq \ell \leq k, 1 \leq i_j \leq n\}$$

is a basis for  $J_{x_0}^k(\mathbb{R}^n, \mathbb{R})_0$ . We use  $J^k(\mathbb{R}^n, \mathbb{R})_0$  to denote the trivial bundle obtained by assigning the vector space  $J_{x_0}^k(\mathbb{R}^n, \mathbb{R})_0$  to each  $x_0 \in \mathbb{R}^n$ .

For any  $f \in C^k(\mathbb{R}^n, \mathbb{R}^m)$  and  $x_0 \in \mathbb{R}^n$ ,  $f$  induces a linear map  $J_{x_0}^2(f) : J_{f(x_0)}^k(\mathbb{R}^m, \mathbb{R})_0 \rightarrow J_{x_0}^k(\mathbb{R}^n, \mathbb{R})_0$  by precomposition. The entries of  $J_{x_0}^2(f)$  depend on the partials of  $f$  at  $x_0$  up to order  $k$ . Moreover, jets behave nicely under composition; precisely,  $J_{x_0}^2(f \circ g) = J_{x_0}^2(g) \cdot J_{g(x_0)}^2(f)$  in the sense of matrix multiplication. Often, we use  $J^k(f)$  to denote the matrix-valued function  $x_0 \mapsto J_{x_0}^k(f)$ . For example, consider a smooth map  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with coordinate functions  $f_1, f_2$ . The 2-jet  $J^2(f)$ , expressed with respect to the bases  $\mathcal{B}_{x_0}$  and  $\mathcal{B}_{f(x_0)}$  on  $J_{x_0}^2(\mathbb{R}^2, \mathbb{R})_0$  and  $J_{f(x_0)}^2(\mathbb{R}^2, \mathbb{R})_0$ , respectively, is

$$J^2(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & 0 & 0 & 0 \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & 0 & 0 & 0 \\ \frac{\partial^2 f_1}{\partial x^2} & \frac{\partial^2 f_2}{\partial x^2} & \left(\frac{\partial f_1}{\partial x}\right)^2 & \left(\frac{\partial f_2}{\partial x}\right)^2 & \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial x} \\ \frac{\partial^2 f_1}{\partial y^2} & \frac{\partial^2 f_2}{\partial y^2} & \left(\frac{\partial f_1}{\partial y}\right)^2 & \left(\frac{\partial f_2}{\partial y}\right)^2 & \frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial y} \\ \frac{\partial^2 f_1}{\partial x \partial y} & \frac{\partial^2 f_2}{\partial x \partial y} & \frac{\partial f_1}{\partial x} \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial x} \frac{\partial f_2}{\partial y} & \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial y} \end{pmatrix}.$$

In general, one has

$$(B.6) \quad \|J_{x_0}^2(f)\|_\infty \leq \max(2\|(Df)_{x_0}\|_\infty^2, \|(D^2f)_{x_0}\|_\infty).$$

### C. COMPUTER ERROR ESTIMATES

In `K3entropy`, `periodicExact.c`, we use the GNU MPFR library to estimate each  $f_i^c(a_i, b_i)$  with high accuracy. The program checks that the output estimates for  $f_i^c(a_i, b_i)$  agree with  $(a_{i+1} \pmod{10}, b_{i+1} \pmod{10})$  up to twenty-nine decimal places, which we saw in Section 3.5 was sufficient to apply Lemma 3.2.

The purpose of this section is to demonstrate that the values of  $f_i^c(a_i, b_i)$  output by the computer are accurate up to significantly more than twenty-nine decimals. To achieve this, we use interval arithmetic to bound the errors produced throughout the computation. First, we recall the basics of floating point arithmetic and the rounding procedures of GNU MPFR.

The GNU MPFR library performs floating-point operations with fixed precision. A floating-point number is a real number of the form

$$(C.1) \quad \pm 2^p \left( \frac{1}{2} + \sum_{i=2}^N a_i 2^{-i} \right)$$

where  $a_i \in \{0, 1\}$ ,  $N$  is the number of bits used in the mantissa, and  $p$  is an integer. The library allows users to specify  $N$  and the allowed range of  $p$ . For fixed  $N$  and range  $[m, M]$ , we call a floating-point number *representable* if it can be written in the form C.1 for  $p \in [m, M]$ . For instance, the smallest positive value of a representable floating-point number is  $2^m \frac{1}{2}$ , and the largest value is  $2^M (1 - \frac{1}{2^N})$ .

Given a real number  $x$  in the range  $[2^m \frac{1}{2}, 2^M (1 - \frac{1}{2^N})]$ , which we refer to as the *allowable range*, let  $[x]$  denote the sum of the first  $N$  terms of its binary expansion. Then  $[x]$  is representable, and

$$(C.2) \quad \begin{aligned} |x - [x]| &\leq 2^p \cdot 2^{-N} \\ &\leq 2 \cdot |x| \cdot 2^{-N} \\ &= |x| \cdot 2^{-N+1}. \end{aligned}$$

Throughout the section, set

$$r := 2^{-N+1}.$$

When the rounding mode ‘MPFR\_RNDZ’ is enabled in MPFR, the built-in MPFR functions perform exact calculations on floating-point numbers and round the result using  $[\cdot]$ . For example, if we use  $+^c$  to denote MPFR addition,

$$x +_* y = [x + y]$$

as long as  $x, y$  are representable and  $x + y$  is in the allowable range. The functions  $\mathcal{D}, p, \sigma_i, f, f_i^c$  are compositions of the elementary MPFR functions  $+_*, -_*, \cdot_*, \sqrt_*, /_*$ , and we use  $[\mathcal{D}], [p], [\sigma_i], [f], [f_i^c]$  to denote their corresponding MPFR implementations. For example, given representable  $x, y$ , we have

$$\begin{aligned} [\mathcal{D}](x, y) &= 100 \cdot_* (x \cdot_* x) \cdot_* (y \cdot_* y) + 8 \cdot_* (1 + x \cdot_* x)(1 + y \cdot_* y) \\ &\quad - 4(1 + x \cdot_* x) \cdot_* (1 + x \cdot_* x) \cdot_* (1 + y \cdot_* y) \cdot_* (1 + y \cdot_* y) \end{aligned}$$

assuming that the intermediate values are in the allowable range. We aim to quantify the error introduced by replacing  $f_i^c$  with  $[f_i^c]$ , that is, to bound

$$\|[f_i^c] - f_i^c\|_2$$

in terms of  $N$ .

Before beginning, recall the following identities of interval arithmetic:

$$\begin{aligned} a \cdot B(x, r_1) &= B(ax, ar_1) \\ a \pm B(x, r_1) &= B(x \pm a, r_1) \\ B(x, r_1) \pm B(y, r_2) &= B(x \pm y, r_1 + r_2) \\ B(x, r_1) \cdot B(y, r_2) &= B(xy, r_1 r_2) \\ \frac{B(x, r_1)}{B(y, r_2)} &\subset B\left(\frac{x}{y}, \frac{|x| + r_1}{(|y| - r_2)^2} r_2\right) \\ \sqrt{B(x, r_1)} &\subset B\left(\sqrt{x}, \frac{1}{2\sqrt{|x| - r_1}} r_1\right) \end{aligned}$$

where in the penultimate line we assume  $|y| - r_2 > 0$ , and in the last line  $|x| - r_1 > 0$ . All of the above expressions have analogs when we instead use elementary MPFR functions. Using C.2, we find

$$\begin{aligned}
a \cdot_* b &\in B(ab, abr) \\
a \cdot_* B(x, r_1) &\subset B(ax, |a|r_1 + (|ax| + |a|r_1)r) \\
a \pm_* B(x, r_1) &\subset B(x \pm a, r_1 + (|x| + r_1)r) \\
B(x, r_1) \pm_* B(y, r_2) &\subset B(x \pm y, r_1 + r_2 + (|x| + |y| + r_1 + r_2)r) \\
B(x, r_1) \cdot_* B(y, r_2) &\subset B(xy, r_1 r_2 + (|xy| + r_1 r_2)r) \\
\frac{B(x, r_1)}{B(y, r_2)} \star &\subset B\left(\frac{x}{y}, \frac{|x| + r_1}{(|y| - r_2)^2} r_2 + \left(\frac{|x|}{|y|} + \frac{|x| + r_1}{(|y| - r_2)^2} r_2\right)r\right) \\
\sqrt{B(x, r_1)} \star &\subset B\left(\sqrt{x}, \frac{1}{2\sqrt{|x| - r_1}} r_1 + \left(\sqrt{x} + \frac{1}{2\sqrt{|x| - r_1}} r_1\right)r\right)
\end{aligned}$$

**Lemma C.3.** *Assume  $N \geq 34$ , i.e.  $r \leq 10^{-10}$ , and the range  $[m, M]$  is sufficiently large. For all representable  $x, y$  in the domain of  $f_i^c$ ,*

$$\| [f_i^c](x, y) - f_i^c(x, y) \|_2 \leq 4 \cdot 10^6 r.$$

For example, if  $N = 500$ , then  $r = 2^{-499}$  and  $\delta_{f^c} \leq 2 \cdot 10^{-144}$ , which is much smaller than  $10^{-30}$ .

*Proof of Lemma C.3.* Our proof proceeds via a series of Claims.

**Claim C.4.** *If  $x, y$  are representable and satisfy  $0 \leq \mathcal{D}(x, y)$ , then  $|\alpha](x, y) - \alpha(x, y)| \leq \delta_\alpha$  where*

$$\delta_\alpha := 2000r.$$

*Proof.* Recall that  $0 \leq \mathcal{D}(x, y)$  implies  $|x|, |y| \leq c < 2.5$  (Lemma B.2). Then

$$\begin{aligned}
\alpha](x, y) &= \frac{x \cdot_* y}{(1 +_* x \cdot_* x) \cdot_* (1 +_* y \cdot_* y)} \star \\
&\in \frac{B(xy, c^2 r)}{(1 +_* B(x^2, c^2 r)) \cdot_* (1 +_* B(y^2, c^2 r))} \star \\
&\subset \frac{B(xy, c^2 r)}{B(1 + x^2, c^2 r + (1 + c^2 + c^2 r)r) \cdot_* B(1 + y^2, c^2 r + (1 + c^2 + c^2 r)r)} \star \\
&\subset \frac{B(xy, 9r)}{B(1 + x^2, 15r) \cdot_* B(1 + y^2, 15r)} \star \\
&\subset \frac{B(xy, 9r)}{B((1 + x^2)(1 + y^2), 2(1 + c^2)(15r) + 100r^2 + ((1 + c^2)^2 + 2(1 + c^2)(15r) + 100r^2)r)} \star \\
&\subset \frac{B(xy, 9r)}{B((1 + x^2)(1 + y^2), 300r)} \star \\
&\subset B(\alpha(x, y), \frac{|xy| + 9r}{(|(1 + x^2)(1 + y^2)| - 300r)^2} 300r) \\
&\subset B(\alpha(x, y), \frac{c^2 + 9r}{(1 - 300r)^2} 300r) \\
&\subset B(\alpha(x, y), 2000r)
\end{aligned}$$

when  $r$  is sufficiently small. □

**Claim C.5.** *If  $x, y$  are representable and satisfy  $0 \leq \mathcal{D}(x, y)$ , then  $|\beta](x, y) - \beta(x, y)| \leq \delta_\beta$  where  $\delta_\beta := 2000r$ .*

*Proof.* The proof is similar to that of Lemma C.4. □

**Claim C.6.** *If  $x, y$  are representable and satisfy  $0 \leq \mathcal{D}(x, y)$ , then  $|\mathcal{D}](x, y) - \mathcal{D}(x, y)| \leq \delta_D$  where  $\delta_D := 10^6 r$ .*

*Proof.* As in Lemma C.4,

$$\begin{aligned}
[\mathcal{D}](x, y) &= [100 \cdot_* ((x \cdot_* x) \cdot_* (y \cdot_* y))] +_* [8 \cdot_* ((1 +_* (x \cdot_* x)) \cdot_* (1 +_* (y \cdot_* y)))] \\
&\quad -_* [4 \cdot_* (((1 +_* (x \cdot_* x)) \cdot_* (1 +_* (x \cdot_* x))) \cdot_* ((1 +_* (y \cdot_* y)) \cdot_* (1 +_* (y \cdot_* y)))))] \\
\text{(C.7)} \quad &\subset [100 \cdot_* (B(x^2, c^2r) \cdot_* B(y^2, c^2r))] +_* [8 \cdot_* ((1 +_* B(x^2, c^2r)) \cdot_* (1 +_* B(y^2, c^2r)))] \\
\text{(C.8)} \quad &-_* [4 \cdot_* (((1 +_* B(x^2, c^2r)) \cdot_* (1 +_* B(x^2, c^2r))) \cdot_* (((1 +_* B(y^2, c^2r)) \cdot_* (1 +_* B(y^2, c^2r)))))]
\end{aligned}$$

The sum C.7 is contained in

$$\begin{aligned}
&[100 \cdot_* B(x^2y^2, 2c^4r + c^4r^2 + r(c^4 + 2c^4r + c^4r^2))] \\
&\quad +_* [8 \cdot_* (B(1 + x^2, c^2r + r(1 + c^2 + c^2r)) \cdot_* B(1 + y^2, c^2r + r(1 + c^2 + c^2r)))] \\
&\subset [100 \cdot_* B(x^2y^2, 243r)] +_* [8 \cdot_* (B(1 + x^2, 10r) \cdot_* B(1 + y^2, 10r))] \\
&\subset B(100x^2y^2, 100(243r) + r(100c^4)) \\
&\quad +_* (8 \cdot_* B((1 + x^2)(1 + y^2), 20r(1 + c^2) + 100r^2 + r((1 + c^2)^2 + 20r(1 + c^2) + 100r^2))) \\
&\subset B(100x^2y^2, 3 \cdot 10^4r) +_* (8 \cdot_* B((1 + x^2)(1 + y^2), 300r)) \\
&\subset B(100x^2y^2, 3 \cdot 10^4r) +_* B(8(1 + x^2)(1 + y^2), 300r + r(8(1 + c^2)^2 + 300r)) \\
&\subset B(100x^2y^2, 3 \cdot 10^4r) +_* B(8(1 + x^2)(1 + y^2), 400r) \\
&\subset B(100x^2y^2 + 8(1 + x^2)(1 + y^2), 3 \cdot 10^4r + 400r + r(100c^4 + 8(1 + c^2)^2 + 3 \cdot 10^5r + 400r)) \\
&\subset B(100x^2y^2 + 8(1 + x^2)(1 + y^2), 3 \cdot 10^4r + 400r + 4500r) \\
&\subset B(100x^2y^2 + 8(1 + x^2)(1 + y^2), 3.5 \cdot 10^4r).
\end{aligned}$$

Moreover, the term C.8 is contained in

$$\begin{aligned}
&4 \cdot_* ((B(1 + x^2, c^2r + r(1 + c^2 + c^2r)) \cdot_* B(1 + x^2, c^2r + r(1 + c^2 + c^2r))) \\
&\quad \cdot_* (B(1 + y^2, c^2r + r(1 + c^2 + c^2r)) \cdot_* B(1 + y^2, c^2r + r(1 + c^2 + c^2r)))) \\
&\subset 4 \cdot_* ((B(1 + x^2, 10r) \cdot_* B(1 + x^2, 10r)) \cdot_* (B(1 + y^2, 10r) \cdot_* B(1 + y^2, 10r))) \\
&\subset 4 \cdot_* (B((1 + x^2)^2, 20r(1 + c^2) + 100r + r((1 + c^2)^2 + 20r(1 + c^2) + 100r^2))) \\
&\quad \cdot_* (B((1 + y^2)^2, 20r(1 + c^2) + 100r + r((1 + c^2)^2 + 20r(1 + c^2) + 100r^2))) \\
&\subset 4 \cdot_* (B((1 + x^2)^2, 300r) \cdot_* B((1 + y^2)^2, 300r)) \\
&\subset 4 \cdot_* B((1 + x^2)^2(1 + y^2)^2, 600r(1 + c^2)^2 + 9 \cdot 10^4r^2 + r((1 + c^2)^4 + 600r(1 + c^2)^2 + 9 \cdot 10^4r^2)) \\
&\subset 4 \cdot_* B((1 + x^2)^2(1 + y^2)^2, 3.5 \cdot 10^4r) \\
&\subset B(4(1 + x^2)^2(1 + y^2)^2, 4 \cdot 3.5 \cdot 10^4r + r(4(1 + c^2)^4 + 4 \cdot 3.5 \cdot 10^4r)) \\
&\subset B(4(1 + x^2)^2(1 + y^2)^2, 1.5 \cdot 10^5r).
\end{aligned}$$

Thus

$$\begin{aligned}
[\mathcal{D}](x, y) &\in B(100x^2y^2 + 8(1 + x^2)(1 + y^2), 3.5 \cdot 10^4r) -_* B(4(1 + x^2)^2(1 + y^2)^2, 1.5 \cdot 10^5r) \\
&\subset B(\mathcal{D}(x, y), 3.5 \cdot 10^4r + 1.5 \cdot 10^5r + r(|\mathcal{D}(x, y)| + 3.5 \cdot 10^4r + 1.5 \cdot 10^5r)) \\
&\subset B(\mathcal{D}(x, y), (3.5 \cdot 10^4 + 1.5 \cdot 10^5 + |\mathcal{D}(x, y)|)r).
\end{aligned}$$

Now Lemma B.2 implies  $|\mathcal{D}(x, y)| \leq 1.5 \cdot 10^4$  so we obtain

$$[\mathcal{D}](x, y) \in B(\mathcal{D}(x, y), 2 \cdot 10^5r),$$

as desired.  $\square$

**Claim C.9.** *If  $x, y$  are representable and satisfy  $1 \leq |\mathcal{D}(x, y)|$ , then  $||p_{\pm}](x, y) - p_{\pm}(x, y)|| \leq \delta_p$  where  $\delta_p := 1.6 \cdot 10^6r$ .*

*Proof.* Let  $C = 1.5 \cdot 10^4$  so that  $|\mathcal{D}(x, y)| \leq C$ . Recall  $p_{\pm} = -5\alpha \pm \frac{1}{2}\beta\mathcal{D}^{0.5}$ . Therefore

$$\begin{aligned}
[p_{\pm}](x, y) &= (-5 \cdot_{\star} [\alpha](x, y)) \pm_{\star} \left( \frac{1}{2} \cdot_{\star} ([\beta](x, y) \cdot_{\star} \sqrt{[\mathcal{D}]}(x, y)_{\star}) \right) \\
&\in -5 \cdot_{\star} B(\alpha(x, y), \delta_{\alpha}) \pm_{\star} \left( \frac{1}{2} \cdot_{\star} (B(\beta(x, y), \delta_{\beta}) \cdot_{\star} \sqrt{B(\mathcal{D}(x, y), \delta_{\mathcal{D}})_{\star}}) \right) \\
&\subset B(-5\alpha(x, y), 5\delta_{\alpha} + r(5|\alpha(x, y)| + 5\delta_{\alpha})) \\
&\pm_{\star} \left( \frac{1}{2} \cdot_{\star} (B(\beta(x, y), \delta_{\beta}) \cdot_{\star} B(\sqrt{\mathcal{D}(x, y)}, \frac{1}{2\sqrt{|\mathcal{D}(x, y)| - \delta_{\mathcal{D}}}}\delta_{\mathcal{D}} + r(\sqrt{\mathcal{D}(x, y)} + \frac{1}{2\sqrt{|\mathcal{D}(x, y)| - \delta_{\mathcal{D}}}}\delta_{\mathcal{D}}))) \right) \\
&\subset B(-5\alpha(x, y), 5\delta_{\alpha} + 6r) \pm_{\star} \left( \frac{1}{2} \cdot_{\star} (B(\beta(x, y), \delta_{\beta}) \cdot_{\star} B(\sqrt{\mathcal{D}(x, y)}, \frac{1}{2\sqrt{1 - \delta_{\mathcal{D}}}}\delta_{\mathcal{D}} + r\sqrt{C})) \right) \\
&\subset B(-5\alpha(x, y), 5\delta_{\alpha} + 6r) \pm_{\star} \left( \frac{1}{2} \cdot_{\star} (B(\beta(x, y), \delta_{\beta}) \cdot_{\star} B(\sqrt{\mathcal{D}(x, y)}, 1.01\delta_{\mathcal{D}} + r\sqrt{C})) \right) \\
&\subset B(-5\alpha(x, y), 5\delta_{\alpha} + 6r) \pm_{\star} \\
&\quad \left[ \frac{1}{2} \cdot_{\star} B \left( \beta(x, y)\sqrt{\mathcal{D}(x, y)}, \delta_{\beta}\sqrt{C} + 1.01\delta_{\mathcal{D}} + r\sqrt{C} + \delta_{\beta}(1.01\delta_{\mathcal{D}} + r\sqrt{C}) \right. \right. \\
&\quad \quad \left. \left. + r(\sqrt{C} + \delta_{\beta}\sqrt{C} + 1.01\delta_{\mathcal{D}} + r\sqrt{C} + \delta_{\beta}(1.01\delta_{\mathcal{D}} + r\sqrt{C})) \right) \right] \\
&\subset B(-5\alpha(x, y), 5\delta_{\alpha} + 6r) \pm_{\star} \left( \frac{1}{2} \cdot_{\star} B(\beta(x, y)\sqrt{\mathcal{D}(x, y)}, \delta_{\beta}\sqrt{C} + 1.01\delta_{\mathcal{D}} + r\sqrt{C} + r(\sqrt{C} + 1)) \right)
\end{aligned}$$

where in several lines we use  $|\beta(x, y)|, |\alpha(x, y)| \leq 1$ . Then from Lemmas C.4, C.5, and C.6 we have

$$\begin{aligned}
[p_{\pm}](x, y) &\in B(-5\alpha(x, y), 2 \cdot 10^5 r) \pm_{\star} \left( \frac{1}{2} \cdot_{\star} B(\beta(x, y)\sqrt{\mathcal{D}(x, y)}, 1.3 \cdot 10^6 r) \right) \\
&\subset B(p_{\pm}(x, y), 2 \cdot 10^5 r + 1.3 \cdot 10^6 r + cr) \\
&\subset B(p_{\pm}(x, y), 1.6 \cdot 10^6 r),
\end{aligned}$$

as desired. In the second line, we used that multiplication by any power of 2 is implemented without rounding whenever the result is in the allowable range.  $\square$

**Claim C.10.** *If  $x, y, z$  are representable and satisfy  $\mathcal{D}(x, y), \mathcal{D}(x, z), \mathcal{D}(y, z) \geq 0$ , then  $\|[\sigma_i](x, y, z) - \sigma_i(x, y, z)\|_2 \leq \delta_{\sigma}$  for each  $1 \leq i \leq 3$ , where*

$$\delta_{\sigma} := 10\delta_{\alpha} + 14r$$

*Proof.* Recall

$$\sigma_1(x, y, z) = (-x - 10\alpha(y, z), y, z).$$

Thus

$$\begin{aligned}
[\sigma_1](x, y, z) &\in (-x -_{\star} (10 \cdot_{\star} B(\alpha(y, z), \delta_{\alpha}))) \times \{y\} \times \{z\} \\
&\subset (-x -_{\star} B(10\alpha(y, z), 10\delta_{\alpha} + r(10 + 10\delta_{\alpha}))) \times \{y\} \times \{z\} \\
&\subset B(-x - 10\alpha(y, z), 10\delta_{\alpha} + r(10 + 10\delta_{\alpha}) + r(c + 10\delta_{\alpha} + r(10 + 10\delta_{\alpha}))) \times \{y\} \times \{z\} \\
&\subset B(-x - 10\alpha(y, z), 10\delta_{\alpha} + 11r + 3r) \times \{y\} \times \{z\},
\end{aligned}$$

concluding the proof.  $\square$

**Claim C.11.** *For all representable  $x, y, z$  satisfying  $\mathcal{D}(x, y), \mathcal{D}(x, z), \mathcal{D}(y, z) \geq 0$ ,  $\|[f](x, y, z) - f(x, y, z)\| \leq \delta_f$  where*

$$\delta_f = \delta_{\sigma}(1 + 10 + (10)^2)$$

*Proof.* Well

$$\delta_f \leq \delta_{\sigma} + \delta_{\sigma} \|D\sigma\|_{C^0(X(\mathbb{R}))} + \delta_{\sigma} \|D\sigma\|_{C^0(X(\mathbb{R}))}^2$$

and we apply  $\|D\sigma_i\|_{X(\mathbb{R}), op} \leq \|D\sigma_i\|_{X(\mathbb{R}), \infty}$  with the derivative bounds from Lemma B.4.  $\square$

**Claim C.12.** *For all representable  $x, y$  satisfying  $(x, y) \in B_{\varepsilon}(0)$ ,  $\|[f_i^c](x, y) - f_i^c(x, y)\|_2 \leq \delta_{f^c}$  where*

$$\delta_{f^c} = \delta_f + 9(10)^3\delta_p$$

*Proof.* Well

$$\begin{aligned} \|[f_i^c](x) - f_i^c(x)\|_2 &\leq \|[f \circ \varphi_i](x) - f \circ \varphi_i(x)\|_2 \\ &\leq \delta_f + \delta_p \|Df\|_{X(\mathbb{R}),op} \end{aligned}$$

and we conclude with Lemma B.4. □

Combining all the claims, we find  $\delta_{fc} \leq 4 \cdot 10^6 r$ , as desired. □

#### D. TABLES

$i$	$\mathcal{D}(a_i, b_i)$	$\partial_1 \mathcal{D}(a_i, b_i)$	$\partial_2 \mathcal{D}(a_i, b_i)$	$\partial_{11} \mathcal{D}(a_i, b_i)$	$\partial_{22} \mathcal{D}(a_i, b_i)$	$\partial_{12} \mathcal{D}(a_i, b_i)$
0	114.24	136.59	-45.98	-419.46	-441.48	-178.78
1	8.71	-22.00	15.52	39.45	13.84	-81.14
2	70.23	86.57	-124.38	-58.60	-2.96	-171.09
3	8.59	-40.85	4.26	113.57	-24.69	-87.82
4	108.31	39.04	162.94	-260.04	-171.83	27.57
5	10.43	23.28	-23.28	28.55	28.55	-92.99
6	108.31	-162.94	-39.04	-171.83	-260.04	27.57
7	8.59	-4.26	-40.85	-24.69	113.57	87.82
8	70.23	124.38	-86.57	-2.96	-58.60	-171.09
9	8.71	15.52	22.00	13.84	39.45	81.14

TABLE 2. Estimates for  $R, M, K$

$i$	$\tilde{L}_i \approx (Df_i^c)_0$
0	$\begin{pmatrix} -0.2159 & -0.3755 \\ -0.4694 & 0.4623 \end{pmatrix}$
1	$\begin{pmatrix} 1.7718 & -2.1227 \\ -12.3247 & 16.3690 \end{pmatrix}$
2	$\begin{pmatrix} 0.3539 & -4.7730 \\ 0.4185 & -4.6570 \end{pmatrix}$
3	$\begin{pmatrix} -3.1432 & -0.4406 \\ -0.0916 & -1.1425 \end{pmatrix}$
4	$\begin{pmatrix} -0.4583 & 0.1150 \\ -0.6163 & 0.8316 \end{pmatrix}$
5	$\begin{pmatrix} 1.4772 & -1.9866 \\ 0.3707 & -2.6806 \end{pmatrix}$
6	$\begin{pmatrix} -0.8852 & 0.0258 \\ -0.1241 & 0.3218 \end{pmatrix}$
7	$\begin{pmatrix} 1.0118 & 1.1967 \\ 13.6472 & 13.3154 \end{pmatrix}$
8	$\begin{pmatrix} -0.6239 & -4.3400 \\ 0.7475 & 5.7641 \end{pmatrix}$
9	$\begin{pmatrix} -1.3601 & 1.6743 \\ -0.3730 & -2.2037 \end{pmatrix}$

TABLE 3. Estimates for  $(Df_i^c)_0$

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